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THE MATHEMATICAL TRAINING OF CHEMISTS[1]

Your committee has endeavored to emphasize ways in which the mathematical training of chemists can be improved. It considers improvement necessary because chemistry is rapidly becoming more mathematical and because adequate preparation becomes more vital when competition for positions is keener. Furthermore it hopes that early in his career the prospective student of chemistry will be told by his advisers of the importance of mathematics, and that the mathematical prerequisites will be definitely stated in the catalogs describing the course.

REQUIREMENTS FOR PHYSICAL CHEMISTRY

The demand for calculus as a universal prerequisite for physical chemistry has been gaining headway rapidly. In 1932 the student of Chemistry, who does not understand the use of calculus, is denied admission to any first class course in physical chemistry. In some cases it may still be necessary to admit students of the biological or agricultural sciences if they agree to study simultaneously the chemically-essential parts of calculus.

The standard courses in mathematics through analytical geometry and differential and integral calculus are sufficient for the first course in physical chemistry. The student of physical chemistry above all should understand fully the meaning and significance of calculus. He should have at immediate command a limited number of formulas (including the differentiation of x, x^n, e^x, a^x, $\log_e x$, uv, u/v, and the integration of x, x^{-1}, x^n, e^x; and the formulas for partial differentiation and for integration between limits) but he need not be ashamed to refer to his book and to a table of integrals for more complicated formulas. Too often the loss of confidence, resulting from a realization that many formulas have been forgotten, is a serious and unnecessary handicap. With this meager equipment the student of physical chemistry can get along, but his progress will depend largely on the adequacy of his mathematical training and insight. His success in the course will be greater, the greater the number of problems in mathematics which he has worked out. He must be thoroughly familiar with algebra and he must be able to make transpositions, solve quadratic equations and handle formulas without hesitation. The successful student of physical chemistry will be quick to translate a phenomenon into mathematical language and he will see clearly and completely the physical significance of his formulas. When properly trained, the student will use his slide rule efficiently for some problems and for others he will use extensive tables of logarithms with unfailing accuracy. He will not hesitate to use the exponential notation for very large

[1] This is a report of the Committee on Mathematics, Division of Chemical Education, American Chemical Society, presented at the Denver meeting, August 24, 1932. In addition to what is here given, the report contained a section of *Suggestions for the Journal of Chemical Education*, and a much longer list of typical problems. The committee consisted of Professor Farrington Daniels, chairman, and Professors George H. Bruce, Charles H. Stone, Walter T. Schrenck, and John H. Yoe.

and small numbers nor will he be disturbed when his graphs have unequal scales along the axes or when they fail to start at zero. He must acquire a sense of the order of accuracy required in different types of problems.

SUGGESTIONS FOR TEACHERS OF MATHEMATICS

Certain parts of the standard courses in mathematics are of more importance to the chemist than other parts. Teachers of mathematics realize this fact and would like to know the particular requirements for students of chemistry. In fact this Committee was appointed as a direct result of a letter from Professor E. R. Hedrick, then president of the American Mathematical Society, stating that suggestions from teachers of chemistry would be welcomed.

A few mathematical operations seem to give particular trouble in physical chemistry and it is hoped that more attention may be paid to them in the preparatory courses. Engineering students with their greater experience in mathematical problems have little trouble. Students can use their logarithm tables but they are sometimes troubled when the logarithm must be handled as a number in a formula, particularly when a negative characteristic is involved. Exponential equations sometimes offer difficulties, also. The idea of partial differentiation is particularly important in physical chemistry and thermodynamics and it is to be hoped that this subject will not be reserved for advanced courses nor placed at the end of a course but that it will receive adequate treatment in the standard courses in calculus.

The chemist usually graphs his data but he would like to fit them with a mathematical formula. This Committee feels that more attention should be paid to curve fitting in analytical geometry. Experience in exact graphical calculations and the finding of areas and tangents would be very welcome also. Chemists would find these things of more value than many theorems in trigonometry and analytical geometry. The compound interest law and the significance of e, the base of natural logarithms, are of special importance.

It is realized that these special needs may not be suitable for engineers and others, who constitute the majority of the classes in mathematics. When possible, an excellent service could be rendered by putting chemistry students into separate sections in the mathematics courses and giving them slightly different material, preferably under an instructor in mathematics who has had some training in the natural sciences.

Whether or not it is practical to put chemists into a separate section, it should be possible to include, along with the problems of interest to engineers and physicists, a few of special interest to chemists. Attached to this report is a list of problems which may be taken as suggestive of the type which chemists would like to see sprinkled through text books on mathematics, as special preparation for physical chemistry.

The committee feels that the responsibility for any action along these lines rests with the chemistry faculties in the colleges and universities. If they take the initiative in presenting their special needs to their department of mathematics, the latter will be glad to coöperate.

REQUIREMENTS FOR ADVANCED PHYSICAL CHEMISTRY

For advanced courses in physical chemistry, and research in the newer fields of physical chemistry the mathematical training through calculus is insufficient. Sooner or later the advanced physical chemist will be blocked by inadequate preparation in mathematics but the distance to which he can go will depend largely on his mathematical ability. The only advice to those who wish to prepare themselves for the new advances in physical chemistry and quantum theory is to go as far as possible with the mastery of advanced mathematics.

It may often be advantageous for a member of the chemistry or mathematics faculty to give a course on selected topics in advanced mathematics for those graduate students in physical chemistry who do not expect to do creative work in these fields.

REQUIREMENTS FOR ELEMENTARY CHEMISTRY

The mathematical requirements for elementary courses in chemistry are not extensive. A knowledge of algebra is sufficient but here again the mathematically-adept student or the person who has worked problems in pure mathematics by the thousands will advance more easily. The concept of ratio and proportion is of course very important (The form $a/b = c/d$ is used in preference to the form $a:b::c:d$).

SELECTED TYPICAL PROBLEMS

1. The dissociation constant K of hydro iodic acid is given by the expression

$$K = \frac{(a - x)(b - x)}{4x^2},$$

where a and b are constants representing the concentration of hydrogen and iodine. What is the value of x in terms of a, b, and K?

2. $10^x = 0.00932$. $x = ?$

3. $e^{-0.04} = ?$

4. $0.700 = 0.02 \log (0.01/x)$. $x = ?$

5. The rate of decay of radon is given by the expression $k = [2.303 \log (I_0/I)]/t$ where I_0 is the radio activity intensity at the beginning of the experiment, I is the intensity after time t, and k is a constant. If the intensity has fallen to one-fifth of its value in 12,800 minutes, what is the value of k? What will be the intensity after 100 minutes? At what time will the intensity have fallen to one-half its value. Solve the preceding problem with the equivalent formula $I = I_0 e^{-kt}$.

6. The vapor pressure P of chloroform at different absolute temperatures T is as follows:

T	293°	303	313	323	333
P	160.5 mm	247.5	369.3	535.0	755.5

Plot log P against $1/T$ and find the equation for the best straight line passing through these points.

7. In a chemical reaction involving two types of molecules the amount of material reacting x in time t is related to the concentrations a and b of the original materials as shown by the relation $dx/dt=k(a-x)(b-x)$ where k is a constant. Integrate and solve for k.

HISTORICAL NOTE ON NEGATIVE NUMBERS[1]

By G. A. MILLER, University of Illinois

Negative numbers now play such an important rôle in mathematics that the question when these numbers began to be used systematically by European mathematicians is of fundamental importance in the history of our science. It may therefore be of interest to note here some widely different views as regards this point which appear in several of our most reliable works of reference. In particular, it is stated in the *Encyklopädie der Mathematischen Wissenschaften*, note 18 of the first article (1898), that the actual calculation with negative numbers begins with R. Descartes (1596–1650) and that he attributed to the same letter sometimes a positive and sometimes a negative value. In the corresponding note (149) of the French edition of this encyclopedia which began to appear in 1904 it is stated, on the contrary, that the systematic calculations with negative numbers are posterior to R. Descartes.

These contradictory statements seem to deserve emphasis here because they were naturally adopted and will probably be adopted in the future by various writers who separately may have consulted only one of these noted authorities. The view that the systematic use of negative numbers is posterior to R. Descartes is now supported by many authorities including J. Tropfke, *Geschichte der Elementar-Mathematik*, volume 2 (1921), page 77, where it is also stated that R. Descartes did not attribute sometimes a positive and sometimes a negative number to the same letter. He did, however, sometimes place a dot before a letter and then this letter with its dot might be replaced by him either by a positive or by a negative number. This is clearly not in strict accord with the statement of the preceding paragraph.

Not only is there a radical difference between the statements relating to the early European use of negative numbers which appear in the two modern mathematical encyclopedias noted above but both of these statements are at variance with the one which appears in a still more modern mathematical encyclopedia; viz., the Italian work entitled *Enciclopedia delle Matematiche Elementari* which began to be published in 1930. In the account of the development of negative numbers found on page 89 of the first part of this work it is stated that while signs of negative numbers appear in the Greek work of Diophantus and in the Indian works by Brahmagupta and Bhaskara, and then in the European works of N. Chuquet and M. Stifel, the conception may be said to be sufficient in H.

[1] Abstract from a lecture before the Mathematical Association of America (Illinois Section), May 6, 1932.

Cardan and is complete in A. Girard. It therefore results that according to this encyclopedia the conception of negative numbers was completed even before the appearance of the famous Géométrie by R. Descartes in 1637.

One of the most striking evidences of the fact that the conception of negative numbers was not fully completed before about the close of the eighteenth century is the view that these numbers are greater than infinity which was expressed even in the second half of the eighteenth century by L. Euler, who was one of the greatest mathematicians of all times and corresponded with leading mathematicians of his day. This view had been announced earlier by the English mathematician, J. Wallis, and was maintained for a long time notwithstanding its absurdity from our point of view. It is also interesting to note in this connection that the real reason which inspired the work of the noted French mathematician, L. N. M. Carnot (1753–1823), along the line of projective geometry was his aversion for negative numbers, which he rejected because he thought that their use led to erroneous conclusions. This was done even in the early part of the nineteenth century. Cf. *Encyclopédie des Sciences Mathématiques*, tome 3, volume 2, page 3.

Some of the contradictory historical statements relating to negative numbers are doubtless due to a failure on the part of various writers to distinguish between the correct practical use of these numbers and a satisfactory explanation of the theory upon which the correct usage should be based. The latter appeared long after the former. Our present object is, however, not the establishment of this late appearance but the exhibition of facts which tend towards an explanation of various contradictory statements relating thereto since they are apt to interfere with the formation of clear views on this important historical question. Fortunately the history of mathematics is becoming a science instead of a chronicle or reportage.

A MINIMUM PROBLEM

By J. V. USPENSKY, Stanford University

1. In the present note we give a solution of the following problem: To find the *best* approximate representation of $(x^2+y^2)^{\frac{1}{2}}$ by a linear function $\lambda x+\mu y$ in a rectangular domain $0 \leqq x \leqq a$; $0 \leqq y \leqq b$. The solution of this problem depends, of course, on the meaning we attach to the expression "the best representation." Here this expression is intended to mean that the deviation of the function $(x^2+y^2)^{\frac{1}{2}}-\lambda x-\mu y$ from 0 in the rectangle $0 \leqq x \leqq a$; $0 \leqq y \leqq b$ should be as small as possible. To avoid misunderstanding we agree to call deviation of a function in a given domain the greatest numerical value the function attains in that domain.

The problem being stated in precise terms, here is the final answer: Without restricting the generality we may assume that $b/a=n \leqq 1$. Then, (i) if $n \geqq (9-\sqrt{17})/8$, parameters λ, μ should be chosen as follows

$$\lambda = \frac{2a - b + (a^2 + b^2)^{1/2}}{3a}$$

$$\mu = \frac{2b - a + (a^2 + b^2)^{1/2}}{3b};$$

and (ii) if $n \leqq (9-\sqrt{17})/8$, we must take

$$\lambda = \frac{4a^2 - 2ab + 2b(ab)^{1/2}}{b^2 + 4a^2}$$

$$\mu = \frac{2a^2 + b^2 - ab - 2a(ab)^{1/2}}{b^2 + 4a^2}.$$

The deviation of $(x^2+y^2)^{1/2}-\lambda x-\mu y$ from 0 in the first case is

$$\delta = [a + b - (a^2 + b^2)^{1/2}]/3$$

whereas in the second case

$$\delta = [ab^2 + 2a^2b - 2(ab)^{3/2}]/(b^2 + 4a^2).$$

2. A few preliminary remarks will greatly facilitate the proof of these statements. The function $\phi(x,y) = (x^2+y^2)^{1/2}-\lambda x-\mu y$ cannot attain its extreme values in an *interior* point of the rectangle. For if the extreme value of $\phi(x, y)$ were attained in an interior point, we would have at this point

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0$$

whence, by virtue of homogeneity, $\phi(x, y) = 0$ which is impossible. Thus the extreme values of $\phi(x, y)$ are attained at points on the boundary. Such points are to be sought only among the following five:

$$x = a,\ y = 0;\ x = 0,\ y = b;\ x = a,\ y = b;$$

$$x = \frac{\lambda b}{(1 - \lambda^2)^{1/2}},\ y = b;\ x = a,\ y = \frac{\mu a}{(1 - \mu^2)^{1/2}}.$$

The fourth and fifth points can be taken into consideration only if we have respectively

$$0 < \lambda < \frac{a}{(a^2 + b^2)^{1/2}} \text{ or } 0 < \mu < \frac{b}{(a^2 + b^2)^{1/2}}.$$

The greatest numerical values of $\phi(x, y)$ can be sought only among the following five numbers:

$$\phi(a, 0) = a(1 - \lambda)$$

$$\phi(0, b) = b(1 - \mu)$$

$$\phi(a, b) = (a^2 + b^2)^{1/2} - \lambda a - \mu b$$

$$\phi\left(\frac{\lambda b}{(1-\lambda^2)^{1/2}}, b\right) = b[(1-\lambda^2)^{1/2} - \mu]$$

$$\phi\left(a, \frac{\mu a}{(1-\mu^2)^{1/2}}\right) = a[(1-\mu^2)^{1/2} - \lambda].$$

The last two numbers can be taken into account only when λ and μ are subject to the limitations mentioned above. Besides, if these numbers are not less in absolute value than the remaining three, they must be *negative* since they correspond to minima of the function $\phi(x, y)$; this can be readily inferred from the fact that

$$\frac{\partial^2\phi}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}} > 0, \quad \frac{\partial^2\phi}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}} > 0.$$

3. Now we can easily dispose of Case 1 in which $n \geqq (9-\sqrt{17})/8$. In this case, taking for λ and μ

$$\lambda = \frac{2 - n + (n^2+1)^{1/2}}{3}$$

$$\mu = \frac{2n - 1 + (n^2+1)^{1/2}}{3},$$

we find that the greatest numerical values of $\phi(x, y)$ are attained only at the points $(0, b)$; $(a, 0)$; (a, b) and

$$\phi(a, 0) = \phi(0, b) = -\phi(a, b) = \frac{a + b - (a^2+b^2)^{1/2}}{3}.$$

To prove this it suffices to notice that of the two inequalities

$$\frac{2 - n + (n^2+1)^{1/2}}{3} \geqq \frac{1}{(n^2+1)^{1/2}}$$

$$\frac{2n - 1 + (n^2+1)^{1/2}}{3n} \geqq \frac{n}{(n^2+1)^{1/2}}$$

the first one is satisfied for every positive $n \leqq 1$ whereas the second requires $n \geqq (9-\sqrt{17})/8$.

It remains to prove that for λ and μ chosen as prescribed, the function $\phi(x, y)$ has the least possible deviation from 0. To this end let $\phi_1(x, y)$ be another function of the same type whose deviation is $\leqq \delta$. The difference $\psi(x, y) = \phi(x, y) - \phi_1(x, y)$ is a linear function $\psi(x, y) = \alpha x + \beta y$ and by hypothesis

$$\psi(a, 0) = \alpha a \geqq 0, \psi(0, b) = \beta b \geqq 0,$$

$$\psi(a, b) = \alpha a + \beta b \leqq 0,$$

which is impossible unless $\alpha = \beta = 0$. Thus Case 1 is disposed of.

4. It is more difficult to dispose of Case 2 when $n \leqq (9-\sqrt{17})/8$. To this end we shall first prove the truth of the following statements:

THEOREM 1. *If* $0<\lambda<1$, *then* $b[\mu-(1-\lambda^2)^{1/2}]\geqq\lambda a+\mu b-(a^2+b^2)^{1/2}$, *the equality sign being possible only when* $\lambda=a/(a^2+b^2)^{1/2}$.

PROOF. From the evident inequality

$$[a-\lambda(a^2+b^2)^{1/2}]^2\geqq 0$$

it follows that

$$b^2(1-\lambda^2)\leqq[-\lambda a+(a^2+b^2)^{1/2}]^2,$$

whence

$$-b(1-\lambda^2)^{1/2}\geqq\lambda a-(a^2+b^2)^{1/2}$$

and

$$b[\mu-(1-\lambda^2)^{1/2}]\geqq\lambda a+\mu b-(a^2+b^2)^{1/2}.$$

THEOREM 2. *The system of equations*

$$a(1-\lambda)=b(1-\mu)=b[\mu-(1-\lambda^2)^{1/2}]$$

has a solution in real numbers λ, μ *such that*

$$\mu>b/(a^2+b^2)^{1/2};\ 0<\lambda\leqq a/(a^2+b^2)^{1/2}$$

provided $n\leqq(9-\sqrt{17})/8$.

PROOF. Putting $1-\mu=\rho$ we have

$$\lambda=1-n\rho,\ 1-2\rho=(1-\lambda^2)^{1/2}$$

and eliminating λ,

$$(4+n^2)\rho^2-2(2+n)\rho+1=0.$$

To satisfy the requirements we take

$$\rho=\frac{n+2-2n^{1/2}}{4+n^2}=\frac{1}{n+2+2n^{1/2}}.$$

In fact the inequality

$$\mu>n/(n^2+1)^{1/2}$$

is equivalent to

$$n+2+2n^{1/2}>n^2+1+n(n^2+1)^{1/2}$$

and it is easy to verify that the last inequality holds for $0<n\leqq 1$. Next

$$\lambda=1-n\rho>0.$$

Finally the condition

$$\lambda\leqq\frac{1}{(n^2+1)^{1/2}}$$

is satisfied if

$$\theta(n)=n+2+2n^{1/2}-2(1+n^{1/2})(n^2+1)^{1/2}\geqq 0.$$

For small positive n

$$\theta(n) > 0$$

and this inequality will hold while n remains less than the least root of the equation $\theta(n)=0$. Setting $n=t^2$ the resulting equation in t is

$$t^2 + 2t + 2 = 2(1 + t)(1 + t^4)^{1/2}$$

whence

$$4t^4 + 8t^3 + 3t^2 - 4t - 4 = 0.$$

This quartic splits into two quadratics

$$2t^2 + 3t + 2 = 0,\ 2t^2 + t - 2 = 0.$$

The first has imaginary roots. The positive root of the second is

$$t = (-1 + \sqrt{17})/4,$$

and correspondingly

$$n = t^2 = (9 - \sqrt{17})/8.$$

Hence $\theta(n) \geqq 0$ if $n \leqq (9-\sqrt{17})/8$.

THEOREM 3. *Numbers* λ *and* μ *being chosen as in Theorem* 2 *we shall have*

$$(a^2 + b^2)^{1/2} - \lambda a - \mu b < 0.$$

PROOF. This inequality is equivalent to

$$(n^2 + 4)(n^2 + 1)^{1/2} < n^3 - n^2 + 4n^{3/2} + 4$$

which, as it is easy to show, holds for $0 < n \leqq 1$.

THEOREM 4. *The function* $\phi(x, y)$ *attains its greatest numerical value bp three times if* λ *and* μ *are chosen as in Theorem* 2 *and* $n \leqq (9-\sqrt{17})/8$.

PROOF. Since

$$\mu > n/(n^2 + 1)^{1/2}$$

the greatest numerical value of $\phi(x, y)$ is either

$$a(1 - \lambda) = b(1 - \mu) = -b\left[(1 - \lambda^2)^{1/2} - \mu\right]$$

or

$$\lambda a + \mu b - (a^2 + b^2)^{1/2}.$$

By Theorems 3 and 1 we have

$$b\left[(1 - \lambda^2)^{1/2} - \mu\right] > \lambda a + \mu b - (a^2 + b^2)^{1/2} > 0;$$

hence the statement is proved.

5. It is easy now to dispose of Case 2. By Theorem 4 the function $\phi(x, y)$ determined as required in Case 2, attains its greatest numerical value three times; namely, for

$$x = a,\ y = 0;\ x = 0,\ y = b;\ x = \lambda b/(1 - \lambda^2)^{1/2},\ y = b;$$

and so that

$$\phi(a, 0) = \phi(0, b) = -\phi[\lambda b/(1 - \lambda^2)^{1/2}, b] = \delta.$$

Let $\phi_1(x, y)$ be another function of the same type as $\phi(x, y)$ whose deviation from 0 is $\leqq \delta$. Then, setting

$$\psi(x, y) = \phi(x, y) - \phi_1(x, y) = \alpha x + \beta y$$

we shall have

$$\psi(a, 0) = \alpha a \geqq 0,\ \psi(0, b) = \beta b \geqq 0$$

$$\psi[\lambda b/(1 - \lambda^2)^{1/2}, b] = \alpha\lambda b/(1 - \lambda^2)^{1/2} + \beta b \leqq 0.$$

But $\lambda > 0$ and hence necessarily $\alpha = \beta = 0$. The proof is now complete.

ON SUMS OF SQUARES[1]

By GORDON PALL, McGill University

1. We shall obtain several results concerning:

1) Numbers which are sums of three positive squares;

2) All numbers which are not sums of s unequal squares, $s = 4$ or 5;

3) A formula for $r_s(n^2c)/r_s(c)$, where $r_s(n)$ denotes the number of representations of n as a sum of s squares;

4) The number of representations of an integer as a sum of s squares with bases in arithmetic progression, $s = 2, 3, 4$;

5) The number $R_s(n)$ of representations of n as a sum of s odd squares, $2 \leqq s \leqq 8$;

6) Representation as a sum of four squares prime to m.

Since the writer duplicated Dubouis' work before noticing the report of it (Dickson's History II, p. 316), it may be well to mention his theorem that, if $s \geqq 6$, the only integers not equal to a sum of s positive squares are

$$1, 2, 3, \cdots, s - 1, \text{ and } s + B,\ (B = 1, 2, 4, 5, 7, 10, 13);$$

that the only $n > 0$ not sums of five positive squares are $1, 2, 3, 4, 5 + B$, and 33; and finally, Descartes' conjecture that the only $n > 0$ not a sum of four positive squares are

$$(1) \qquad 1, 3, 5, 9, 11, 17, 29, 41, 2 \cdot 4^h, 6 \cdot 4^h, 14 \cdot 4^h\ (h \geqq 0).$$

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[1] Most of this paper was written by the author as a National Research Fellow at the California Institute of Technology.

It is trivial, since they are not sums of two squares, that every positive $8n+3$ and $8n+6$ is a sum of three positive squares.[1] It is conjectured that every $2(8n+1)$ except 2 and every $8n+1$ except 1 and 25 is a sum of three positive squares. The last conjecture was verified for every $8n+1$ to 1521.

A. Hurwitz [*Dickson's History* II, p. 271] proved that the only squares not a sum of three positive squares are 4^h and $25\cdot 4^h$. We readily generalize this in the following theorem.

THEOREM 1. *Except for the numbers* $25\cdot 4^h$, *every positive integer which is a sum of three squares at all, and possesses an odd square factor* >1, *is a sum of three positive squares.*

2. The proof depends on the case $s=3$ of the following formula, which is valid in case

$$s = 2, 3, 4, 5, 6, 7, 8, s = 10 \text{ if } c \neq S_2, s = 11 \text{ if } c \neq S_3, \tag{2}$$

$$s = 12 \text{ if } c \text{ is even},$$

where S_r denotes a sum of r squares. In each case (2), we have[2]

$$\frac{r_s(cn^2)}{r_s(c)} = \frac{\psi_2(cn^2)}{\psi_2(c)} \cdot \frac{\Pi\psi_p(cn^2)}{\Pi\psi_p(c)}, \tag{3}$$

the products extending to all odd primes p.

To define $\psi_q(n)\,[=\psi_{q,s}(n)]$ for any prime q and positive integers s and n, write

$$n = 2^\alpha m = r^2 t,\ m \text{ odd},\ t \text{ simple}^3,\ \alpha \geqq 0. \tag{4}$$

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[1] The reader will have no difficulty in proving Descartes' conjecture concerning (1) by using the following classical result, which was first stated in full by Fermat, and was first proved by Legendre in 1798. A positive integer is a sum of three squares if and only if it is not of the form $4^h(8n+7)$ where h and n are integers $\geqq 0$. (Cf. Dickson's History, II, p. ix and p. 302.)

[2] For $s=5$, 7, and for $s=11$ if $c\neq S_3$, formula (3) is proved in the Journal of the London Mathematical Society, vol. 5, (1930), 102–105; for $s=3$ it has appeared in many forms, for example as the Gauss Problem (Cf. *Dickson's History* III, p. 95).

To extend it to the even values s I have merely taken well-known formulas for $r_s(n)$ (Dickson's History, II, pp. 233, 285, 305 ff) and united the resulting expressions for the left member of (3). Since we require $r_2(n)$ in §3 I state a formula for it: write $n=2^\alpha p_1^{a_1}\cdots p_r^{a_r}$ where the p_i are distinct odd primes, $\alpha\geqq 0$, $a_i\geqq 0$, and $r\geqq 0$; then

$$r_2(n) = 4\prod_i \phi(p_i^{a_i}),$$

where

$$\begin{aligned}\phi(p^a) &= a+1 \text{ if } p\equiv 1 \pmod 4;\\ &= 1 \text{ if } p\equiv 3 \text{ and } a \text{ even};\\ &= 0 \text{ if } p\equiv 3 \text{ and } a \text{ odd}.\end{aligned}$$

The reader may deduce (3) for the case $s=2$.

[3] A positive integer without square factors >1 is called *simple*.

Let $a(=a_p)$ denote the exponent to which an odd prime p divides n, and let b, β, σ denote the greatest integers $\leqq \frac{1}{2}a, \frac{1}{2}\alpha, \frac{1}{2}s$ respectively. For example, b is the exponent to which p divides r. For any positive integers s and n, let

$$\zeta_s(n) = \sum_{n=d\delta} d^s, \quad \rho_s(n) = \sum_{n=d\delta}(-1)^{(\delta-1)/2} d^s, \tag{5}$$

where, in the second summation, δ is odd. Then we define

$$\begin{aligned}
\psi_2(n) &= 1 \quad (\alpha = 0,\ s \equiv 0, \bmod 4),\\
&= 2^{\alpha(\sigma-1)} + (-1)^{\sigma/2}\{\zeta_{\sigma-1}(2^{\alpha-1}) - 2\} \quad (\alpha > 0,\ s \equiv 0),\\
&= 2^{\alpha(\sigma-1)} + (-1)^{(m-\sigma)/2} 2^{-(\sigma-1)} \quad (s \equiv 2),\\
&= \left\{1 + (2 \mid s)\frac{2^{(s-3)/2}}{2^{s-2} - 1}\right\}(2^{(s-2)\beta} - 1) + 1 + k \quad (s \text{ odd}),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
k &= (2 \mid s)2^{-\sigma} + (-1)^{(m-s)/4} 2^{-(s-2)} \text{ if } t \equiv s \pmod 4,\\
&= -(2 \mid s)2^{-\sigma} \text{ otherwise;}
\end{aligned}$$

$$\begin{aligned}
\psi_p(n) &= \zeta_{\sigma-1}(p^a) \quad (s \equiv 0),\\
&= \rho_{\sigma-1}(p^a) \quad (s \equiv 2),\\
&= \zeta_{s-2}(p^b) - (\{-1\}^{\sigma} t \mid p) p^{(s-3)/2} \zeta_{s-2}(p^{b-1}) \quad (s \text{ odd}).
\end{aligned} \tag{7}$$

Thus, if s is even, $\psi_p(n) = 1$ unless $p \mid n$;[1] if s is odd, $\psi_p(n) = 1$ unless $p^2 \mid n$.

3. Proof of Theorem 1.

Let p be an odd prime and c simple. As a particular case of (3) we have then

$$r_3(p^2c) = r_3(c)\{p + 1 - (-c \mid p)\}. \tag{8}$$

Now $r_2(p^2c) = \lambda r_2(c)$ where λ is 1, 2, or 3 according as $p \equiv 3$, $p \equiv 1$ and $p \mid c$, or $p \equiv 1$ and $p \nmid c$, the congruences being mod 4. Hence, if $r^*(n)$ denotes for the moment the number of representations of n as a sum of three squares one of which is zero,

$$\begin{aligned}
r^*(p^2c) &= 6\{3 + 2(-1 \mid p)\} \text{ if } c = 1,\\
&= 3\lambda r_2(c) \text{ if } c > 1.
\end{aligned} \tag{9}$$

Hence

$$r_3(p^2) - r^*(p^2) = 6p - 12 - 18(-1 \mid p) > 0$$

except for $p = 5$, when it is zero. If $c > 1$, then $r_3(c) \geqq 3r_2(c)$, and

$$r_3(p^2c) - r^*(p^2c) = r_3(c)\{p + 1 - (-c \mid p)\} - 3\lambda r_2(c) > 0,$$

if $r_3(c) > 0$, for every $p \geqq 3$.

---

[1] The symbols $p \mid n$ and $p \nmid n$ indicate respectively that p divides n and p does not divide n. The symbol $(a \mid b)$ is the Legendre-Jacobi symbol: for example, $(a \mid b) = 0$ if a and b have a g.c.d. > 1; $(-1 \mid p) = 1$ if p is a prime of the form $4n+1$, and $(-1 \mid p) = -1$ if p is a prime $4n+3$.

Theorem 1 evidently follows.

4. The simplest and most elegant special case of (3) is as follows.

Let n be simple and odd. Then for $s=3$, 5, 7, or $s=11$ if $n\equiv 7 \pmod 8$, and for every integer $h\geqq 0$,

$$\frac{r_s(n^{2h+1})}{r_s(n)} = \zeta_{s-2}(n^h). \tag{10}$$

For example, $r_{11}(343)=2^{10}\cdot 3^2\cdot 5\cdot 11\cdot 43\cdot 117307$.

5. THEOREM 2. *The only integers $n>0$ not sums of four unequal squares $\geqq 0$ are $4^h a$, where $h=0, 1, 2, \cdots$, and*

$$\begin{aligned} a = {} & 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 31, 33, 37, 43, 47, 55, 67, 73, \\ & 97, 103, 2, 6, 10, 18, 22, 34, 58, 82. \end{aligned} \tag{11}$$

Let A denote a positive odd integer. The equations

$$\begin{aligned} X &= x+y+z+w, \quad Y = x+y-z-w, \\ Z &= x-y+z-w, \quad W = x-y-z+w, \end{aligned} \tag{12}$$

define a (1, 1) correspondence between the sets of integers x, y, z, w satisfying

$$A = x^2+y^2+z^2+w^2 \tag{13}$$

and the sets of integers X, Y, Z, W satisfying

$$4A = X^2+Y^2+Z^2+W^2,\ X+Y+Z+W \equiv 0 \pmod 4;\ X, Y, Z, W \text{ odd}. \tag{14}$$

Similarly, a (1, 1) correspondence between the integral solutions of (13) and those of

$$2A = s^2+t^2+u^2+v^2,\ s\equiv t \not\equiv u \equiv v \pmod 2 \tag{15}$$

is defined by the equations

$$s = x+y,\ t = x-y,\ u = z+w,\ v = z-w. \tag{16}$$

As immediate corollaries of these facts we have

LEMMA 1. *An odd integer A is a sum of four unequal squares if and only if $4A$ is a sum of four unequal odd squares.*

LEMMA 2. *An odd integer A is a sum of four positive squares if and only if $2A$ is a sum of four unequal squares.*

Now, if either $x^2=y^2$ or $z^2=w^2$ in (13), then

$$stuv = 0. \tag{17}$$

And, if either $x^2=z^2$ or w^2, or $y^2=z^2$ or w^2, then

$$e_1 s + e_2 t + e_3 u + e_4 v = 0, \tag{18}$$

where the $e_i=+1$ or -1. Hence we have

LEMMA 3. *If $2A$ possesses a representation*

$$2A = s^2 + t^2 + u^2 + v^2,\ s, t, u, v \neq 0,\ s^2 > 3A/2, \tag{19}$$

then A is a sum of four unequal squares.

For, the inequality implies the last step of

$$(|t| + |u| + |v|)^2 \leqq 3(t^2 + u^2 + v^2) = 3(2A - s^2) < s^2, \tag{20}$$

which makes (18) impossible.

In proving theorem 2 we can, by the first lemma, restrict ourselves to $a \not\equiv 0 \pmod 4$. Lemma 2 and (1) give the complete results for $a \equiv 2 \pmod 4$.

For A odd (especially $A \equiv 3 \pmod 4$) we shall use lemma 3. Now $2A - s^2 \equiv 6 \pmod 8$ for one of two consecutive even values s. Hence for one of four consecutive choices s, $2A - s^2$ is an S_3 but not an S_2. But four consecutive squares lie between $1\frac{1}{2}A$ and $2A$ if $(2A)^{1/2} - (3A/2)^{1/2} > 4$, $A > 421$.

If $A \equiv 1 \pmod 4$ a different method gives a lower limit than 421. We can choose x^2 from 1, 9, 25, and 49 so that $(A - x^2)/4 \equiv 5 \pmod 8$ if $A \equiv 5 \pmod 8$, and from $1, 9, \cdots, 225$ so that $(A - x^2)/4 \equiv 10$ or $14 \pmod{16}$ if $A \equiv 1 \pmod 8$. In either case $(A - x^2)/4$ is an S_3 but not of the form $\xi^2 + 2\eta^2$. Hence every $8v + 1 \geqq 225$ and every $8v + 5 \geqq 53$ is a sum of four unequal squares. Examination of the few values A below these limits easily gives (11).

In determining all numbers not sums of five unequal squares, of which process we need not give the details, it was convenient to use both the lists (1) and (11), and lemmas 1 and 2 again.

THEOREM 3. *The only integers $n > 0$ not sums of five unequal squares are*:

(21) $1, \cdots, 29, 31, \cdots, 38, 40, \cdots, 45, 47, 48, 49, 52, 53, 56, 58, \cdots, 61, 64,$
$67, 68, 69, 72, 73, 76, 77, 80, 83, 89, 92, 96, 97, 101, 104, 108, 112, 124, 128,$
$136, 137, 188, 224.$

6. We denote $N[n = x_1^2 + \cdots + x_s^2]$ by $r_s(n)$, and

$$N[n = x_1^2 + \cdots + x_s^2;\ \text{all } x_i \text{ odd}] \text{ by } R_s(n).$$

Hence $R_s(n) = 0$ unless $n \equiv s \pmod 8$. We easily prove that: if $n \equiv s \pmod 8$ and $4 \leqq s \leqq 7$, then

$$r_s(n) = c_s R_s(n), \tag{22}$$

where $c_4 = 3/2$, $c_5 = 7/2$, $c_6 = 17/2$, $c_7 = 37/2$. Also, if $n = 2^\alpha m$, $\alpha \geqq 3$, m odd, then

$$R_8(n) = 2^{3\alpha - 1}\zeta_3(m), \tag{23}$$

the right member being an evaluation, from classical formulae, of $[r_8(n) - r_8(n/4)]/36$.

PROOF: By Jacobi's theorem for four squares,

$$N[n = x_1^2 + x_2^2 + x_3^2 + x_4^2; \text{ all } x_i \text{ even}] = \tfrac{1}{2}N[n = x_1^2 + x_2^2 + x_3^2 + x_4^2; \text{ all } x_i \text{ odd}]$$

if $n \equiv 4 \pmod 8$. Hence, if $4 \leqq s \leqq 7$, and $n \equiv s \pmod 8$,

$$r_s(n) = R_s(n) + \binom{s}{4} N(n = x_1^2 + \cdots + x_s^2; C)$$

$$= R_s(n) + \tfrac{1}{2}\binom{s}{4} R_s(n),$$

wherein C denotes the condition $x_1, \cdots, x_4$ even, $x_1^2 + \cdots + x_4^2 \equiv 4 \pmod 8$. This gives (22). Also, by a similar argument,

$$r_8(n) = R_8(n) + \tfrac{1}{2}\binom{8}{4} R_8(n) + N[n = x_1^2 + \cdots + x_8^2; \text{ all } x_i \text{ even}].$$

It may be observed, as a corollary, that $r_5(8n+5)$ is divisible by 7, $r_6(8n+6)$ by 17, $r_7(8n+7)$ by 37.

It is interesting to note that as a consequence of (22) and classical formulae for $r_s(n)$, we have

$$R_s(n) = \frac{32n^{(s-1)/2}}{\pi^{(s-1)/2}} \sum \left(\frac{(-1)^{(s-1)/2}n}{m}\right) \frac{1}{m^{(s-1)/2}} \tag{24}$$

if $n \equiv s \pmod 8$, when $s = 3$, 5 or 7. The summation is over all positive integers m prime to $2n$.

A curious relation is

$$\begin{aligned} &N[24n + 4 = x_1^2 + \cdots + x_4^2; \text{ all } x_i \text{ prime to } 3] \\ &= N[24n + 4 = x_1^2 + \cdots + x_4^2; \text{ all } x_i \text{ prime to } 2] \\ &= 16\zeta_1(6n + 1). \end{aligned} \tag{25}$$

8. Partial results on sums of three or four squares.

Every $40q+3$ from 3 to 3800, except 43, 403, 763, and 3763, is a sum of three squares of the form $(10n+1)^2$. Every $40q+27$ from 1 to 1227, except 427 and 667, is a sum of three squares of the form $(10n+3)^2$.

It is known that every $3q+4 \geqq 4$ is a sum of four squares prime to 3. For no value r is it true that every sufficiently large number $\equiv r \pmod 7$ is a sum of four squares prime to 7 (compare the residues of (1) modulo 7). I have no proof that every $11n > 11$ is a sum of four squares prime to 11. Since 2.4^h and 6.4^h represent the residues $r = 1, 2, 3, 4 \pmod 5$, infinitely many numbers of each progression $5n+r$ are not expressible as a sum of four squares prime to 5. Hence it is interesting to show that every $5n$ except 5 is a sum of four squares prime to 5.

First we consider the possible residues modulo 5 of x^2, y^2, z^2 in the equation

$$5n - w^2 = x^2 + y^2 + z^2, \tag{26}$$

where $5 \dagger w$. Unless two of x, y, z, are zero, which we may suppose not to be the case unless $5n - w^2 = 4^h$ or 25.4^h (by Hurwitz's part of Theorem 1), it is plain that none or two of x, y, z are divisible by 5, and that in the latter case we may assume that $5 \mid y$, $5 \mid z$, $y^2 + z^2 > 0$. By a simple transformation such as $(4Y \pm 3Z)^2 + (3Y \mp 4Z)^2$, possibly repeated, it is easy to replace $(5Y)^2 + (5Z)^2$ by an equal sum of two squares prime to 5.

Hence it remains only to show that, except when $n = 1$, we can choose w prime to 5 so that $5n - w^2$ possesses a representation (26) with no two of x, y, z equal to zero. This will be effected if $5n - w^2$ is of neither of the forms $4^h(8v+7)$ nor 4^h, and is positive. Hence $w = 1$ is effective if $n \equiv 2, 3, 4, 6$, or 7 (mod 8); $w = 2$ if $n \equiv 1, 2, 3, 5$, or 6 (mod 8) but not $n = 1$; $w = 2^{\alpha+1}$ if $n = 2^{2\alpha+1}N$, N odd, $\alpha \geqq 0$; $w = 2^{\alpha-1}$ if $n = 2^{2\alpha}N$, $\alpha \geqq 1$.

9. Let m and n satisfy $0 \leqq n \leqq m$, and either

$$m, n \text{ are odd and relative prime;} \tag{27}$$

or

$$m/2, n/2 \text{ are relative-prime integers of opposite parities.} \tag{28}$$

Let $\rho_s(u)$ where u is any integer $\geqq 0$, denote the number of sets $(x_1, \cdots, x_s)$ of integers x_i satisfying

$$8mu + sn^2 = (2mx_1 + n)^2 + \cdots + (2mx_s + n)^2. \tag{29}$$

There is no loss of generality in assuming (27) or (28), since any sum of s squares whose bases are integers in an arithmetic progression may be reduced essentially to one of those two forms.

PROBLEM. *To express $\rho_s(u)$ by means of numbers of representations in forms with no conditions on the variables.*

We shall succeed in solving this problem only in certain cases where m is *primary*, that is, m is a power of a single prime; or where m is the double of an odd primary number. In the latter case let us call m *secondary*.

First let $s = 2$. Make the transformation $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$. Straightway $\rho_2(u)$ is equal to the number of solutions (y_1, y_2) in integers of

$$4mu + n^2 = (my_1 + n)^2 + m^2y_2^2, \; y_1 \equiv y_2 \pmod 2. \tag{30}$$

Consider then $4mu + n^2 = x^2 + m^2y^2$, in integral (x, y), in the two cases

$$\begin{aligned} &m \text{ primary, } m > 1, \text{ with condition (27); } m \text{ secondary, } m > 2, \\ &\qquad \text{with condition (28).} \end{aligned} \tag{31}$$

From $x^2 \equiv n^2 \pmod{m}$ follows $\pm x \equiv n \pmod{m}$ for just one sign of $\pm x$, since otherwise $2n$ would be divisible by an odd prime factor of m. In all cases $(\pm x - n)/m \equiv y \pmod 2$. This proves that when (31) holds,

$$\rho_2(u) = \tfrac{1}{2}N[4mu + n^2 = x^2 + m^2y^2]. \tag{32}$$

Simpler formulae are easily found for $m = 1$, 2, and 4.

For $s = 3$ we use the transformation

$$\begin{gathered} 3x_1 = y_1 - y_2,\ 6x_2 = 2y_1 + y_2 - 3y_3,\ 6x_3 = 2y_1 + y_2 + 3y_3, \\ y_1 \equiv y_2 \pmod 3,\ y_2 \equiv y_3 \pmod 2, \end{gathered} \tag{33}$$

whence integers x_i correspond to integers y_i and conversely. Hence $\rho_3(u)$ is the number of solutions (y_1, y_2, y_3) of

$$\begin{gathered} 24mu + 9n^2 = (2my_1 + 3n)^2 + 2m^2y_2^2 + 6m^2y_3^2, \\ y_1 \equiv y_2 \pmod 3,\ y_2 \equiv y_3 \pmod 2. \end{gathered} \tag{34}$$

Now in each of the three cases

$$\begin{gathered} 3 \nmid m,\ m \text{ primary, with condition (27)}; \\ 3 \nmid m,\ m \text{ secondary, with condition (28)}; \\ m = 2^{\alpha+2},\ \alpha \geqq 0, \text{ with condition (28)}; \end{gathered} \tag{35}$$

the integers x, y, z satisfying

$$24mu + 9n^2 = x^2 + 2m^2y^2 + 6m^2z^2 \tag{36}$$

will satisfy, for every integer $u \geqq 0$, the condition $x \equiv 3n \pmod{2m}$ by, and only by, a choice of sign of x; the condition $y \equiv z \pmod 2$ automatically; and the condition $x \equiv 3n + 2my \pmod{6m}$ automatically if $3 \mid y$ but only by choice of sign of y if $3 \nmid y$. Hence in cases (35),

$$\begin{aligned} \rho_3(u) = {} & \tfrac{1}{4}N[8mu + 3n^2 = 3x^2 + 6m^2y^2 + 2m^2z^2] \\ & + \tfrac{1}{4}N[24mu + 9n^2 = x^2 + 2m^2y^2 + 6m^2z^2]. \end{aligned} \tag{37}$$

Simpler like formulae hold if $m = 1$, 2, 3, or 6.

For $s = 4$ we apply the transformation

$$\begin{gathered} x_1 + x_2 + x_3 + x_4 = y_1,\ x_1 + x_2 - x_3 - x_4 = y_2, \\ x_1 - x_2 + x_3 - x_4 = y_3,\ x_1 - x_2 - x_3 + x_4 = y_4, \\ y_1 \equiv y_2 \equiv y_3 \equiv y_4 \pmod 2,\ y_1 + y_2 + y_3 + y_4 \equiv 0 \pmod 4. \end{gathered} \tag{38}$$

Hence $\rho_4(u)$ is the number of integers sets $(y_1, \cdots, y_4)$ satisfying

$$\begin{gathered} 8mu + 4n^2 = (my_1 + 2n)^2 + m^2(y_2^2 + y_3^2 + y_4^2), \\ \sum y_i \equiv 0 \pmod 4,\ y_2 \equiv y_3 \equiv y_4 \pmod 2. \end{gathered} \tag{39}$$

For brevity write $\phi_m(k) = N[k = x^2+m^2y^2+m^2z^2+m^2t^2]$. Then we easily see from (39) that, if m is primary and (27) holds,

$$\rho_4(u) = \tfrac{1}{4}\phi_m(8mu + 4n^2) + \tfrac{1}{4}\phi_m(2mu + n^2); \tag{40}$$

that if $m = 2M$, M primary, and (28) holds,

$$\rho_4(u) = \tfrac{1}{4}\phi_M(2mu + n^2) + \tfrac{1}{4}(-1)^n\phi_M(Mu + \tfrac{1}{4}n^2); \tag{41}$$

that, if u is odd, $m = 4M$, $n = 2N$, M primary, N odd, and M prime to N, then

$$\rho_4(u) = \tfrac{1}{4}N[2Mu + N^2 = 4x^2 + M^2(y^2 + z^2 + t^2)]. \tag{42}$$

Simpler formulae are easily obtained for $m = 1, 2, 4$.

A NEW BOUND FOR THE ZEROS OF POLYNOMIALS

By E. C. WESTERFIELD, University of Colorado

In the treatment of the equation with complex coefficients

$$z^n = \sum_1^n b_r z^{n-r}, \qquad b_r \text{ complex}, \tag{1}$$

it is convenient to let p_r represent the absolute value of b_r, and in the present paper it is necessary to adopt the convention that q_i represent the positive quantities $p_r^{1/r}$ arranged in order of decreasing magnitude.

Following this notation, a simple bound due to Carmichael[1] is written

$$|z| \leqq \sum_1^n q_i, \tag{a}$$

one obtained by Fujiwara[2] becomes

$$|z| < 2q_1, \tag{b}$$

while one due to the present author[3] becomes

$$|z| \leqq q_1 + q_2. \tag{c}$$

Comparison of the three bounds given above led the present author to form the expression

$$|z| \leqq \sum_1^n g_i q_i, \tag{d}$$

and to attempt to determine the optimum set of values for g_i for which (d) would always be true. The following theorem resulted:

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[1] Bulletin of the American Mathematical Society, Vol. 24 (1917–18), pp. 286–296.
[2] Tôhoku Mathematical Journal, Vol. 10 (1916), pp. 167–171.
[3] The American Mathematical Monthly, Vol. 38 (1931), pp. 30–35.

Theorem: *Every root of equation* (1) *must satisfy inequality* (*d*) *when the positive quantities* g_i *are obtained through the equations*

$$y_m = \sum_1^m g_i, \qquad m = 1, \cdots, n, \tag{2}$$

the positive quantities y_m *being determined by the equations*

$$y_m^m = \sum_1^m y_m^{m-r}, \qquad m = 1, \cdots, n. \tag{3}$$

For the proof of this theorem we employ the following previously developed lemmas:[1]

Lemma 1: *Any value of x satisfying the inequality*

$$x^n \leqq \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0,$$

will form a lower bound for the sole positive root of the positive term equation

$$x^n = \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0. \tag{4}$$

Lemma 2: *Any positive value of x satisfying the inequality*

$$x^n \geqq \sum_1^n p_r x^{n-r}, \qquad p_r \geqq 0,$$

will form an upper bound for the sole positive root of (4).

Lemma 3: *No root of equation* (1) *can be greater in absolute value than the positive root of* (4) *when the coefficients of* (1) *and* (4) *are related through the equalities*

$$p_r = |b_r| \qquad r = 1, \cdots, n.$$

From Lemma 3 we see that in order to prove our theorem, we need only show that the positive root of (4) must satisfy the inequality

$$x \leqq \sum_1^n g_i q_i, \tag{5}$$

where the q_i are simply the positive expressions $p_r^{1/r}$ arranged in order of decreasing magnitude.

Employing a previously developed inequality[2] for the positive root of (4),

$$q_1 \leqq x < 2q_1,$$

[1] See the author's paper, loc. cit.

[2] See the author's paper, loc. cit.

we see that the positive roots of equations (3) must satisfy the inequalities

$$1 \leqq y_m < 2, \qquad m = 1, \cdots, n. \tag{6}$$

Choosing one of equations (3) and multiplying it through by y_m^{n-m} we have

$$y_m^n = \sum_1^m y_m^{n-r}, \qquad 0 < m \leqq n. \tag{7}$$

Since y_m is not less than 1, decreasing the degree of any term in the right member of (7) will not increase its value. This gives us

$$y_m^n \geqq \sum_1^m y_m^{n-k_r}, \qquad r \leqq k_r \leqq n;$$

and, applying Lemma 2, we see that the positive root of the $(m+1)$-nomial equation

$$y^n = \sum_1^m y^{n-k_r}, \qquad r \leqq k_r \leqq n,$$

must satisfy the inequality

$$y \leqq y_m \qquad 0 < m \leqq n.$$

Replacing y by x/a, where a is an arbitrary positive quantity and multiplying through by a^n, we see that the positive root of the $(m+1)$-nomial equation

$$x^n = \sum_1^m a^{k_r} x^{n-k_r}, \qquad r \leqq k_r \leqq n,$$

must satisfy the inequality

$$x = ay \leqq ay_m = a \sum_1^m g_i \equiv \sum_1^n g_i q_i.$$

This is identical with inequality (5), and our theorem is proven for the restricted class of equations in which q_i are each equal to either zero or an arbitrary positive quantity a.

We will now show that if our theorem is true when each of the q_i is equal to either zero or any one of $s-1$ arbitrary positive quantities, $1<s\leqq n$, then it is true when each of them is equal to either zero or any one of s arbitrary positive quantities. If we can do this our theorem is proven, since we have already shown it for one arbitrary positive quantity or zero.

In the equation

$$x^n = \sum_1^n q_i^{k_i} x^{n-k_i}, \tag{8}$$

where the k_i are simply the integers 1 to n inclusive arranged in any order whatsoever, we will take the q_i to satisfy the following relations:

$$
\begin{aligned}
&q_i = C + ew, && i = 1, \cdots, m_1;\\
&q_i = C - (C - A_{s-2})w, && i = m_1 + 1, \cdots, m_2;\\
&q_i = A_{s-2}, && i = m_2 + 1, \cdots, m_3;\\
&q_i = A_{s-3}, && i = m_3 + 1, \cdots, m_4;\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&q_i = A_1, && i = m_{s-1} + 1, \cdots, m_s;\\
&q_i = A_0 \equiv 0, && i = m_s + 1, \cdots, n,
\end{aligned}
\tag{9}
$$

where e is a positive quantity to be evaluated later, and where

$$
\begin{aligned}
&0 < A_1 < \cdots < A_{s-2} < C,\\
&1 \leqq m_1 < \cdots < m_s \leqq n,\\
&0 \leqq w \leqq 1,\\
&1 < s \leqq n.
\end{aligned}
\tag{10}
$$

If our theorem applies to this case, the positive root of (8) must satisfy inequality (5). Remembering the expression for y_m given by (2), this gives us

$$
\begin{aligned}
x \leqq (C + ew)y_{m_1} + (C - Cw + A_{s-2}w)(y_{m_2} - y_{m_1})\\
+ A_{s-2}(y_{m_3} - y_{m_2}) + \cdots + A_1(y_{m_s} - y_{m_{s-1}}).
\end{aligned}
\tag{11}
$$

We may now choose the arbitrary positive quantity e to satisfy the equation

$$
ey_{m_1} = (C - A_{s-2})(y_{m_2} - y_{m_1}),
\tag{12}
$$

so as to make the coefficient of w in (11) vanish. This gives us

$$
x \leqq Cy_{m_2} + A_{s-2}(y_{m_3} - y_{m_2}) + \cdots + A_1(y_{m_s} - y_{m_{s-1}}) = h,
\tag{13}
$$

which, according to Lemmas 1 and 2, is equivalent to saying that h satisfies the inequality

$$
h^n \geqq \sum_1^n q_i^{k_i} h^{n-k_i} \equiv F(w), \qquad 0 \leqq w \leqq 1.
\tag{14}
$$

By assumption, our theorem holds for $w=0$ and $w=1$, since for each of these values of w, scheme (9) reduces to $s-1$ arbitrary positive quantities and zero. It follows that the inequalities

$$
h^n \geqq F(0), \quad h^n \geqq F(1)
$$

must hold, and investigation shows that the second derivative of $F(w)$ is positive for values of w between 0 and 1; whence inequality (14) must be true and our theorem must hold for w within this range. Suppose, for instance, we set

$w=c$, where c is an arbitrary positive quantity less than 1, we may then write

$$\begin{aligned} C + ec &\equiv A_s, & e &> 0; \\ C - (C - A_{s-2})c &\equiv A_{s-1}, & 0 &< c < 1, \end{aligned} \tag{15}$$

where the quantities A_s and A_{s-1} thus defined are seen to satisfy the conditions

$$A_{s-2} < A_{s-1} < A_s. \tag{16}$$

We see, then, that our theorem is satisfied when the q_i are given by the relations

$$\begin{aligned} q_i &= A_s, && i = 1, \cdots, m_1; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_i &= A_1, && i = m_{s-1} + 1, \cdots, m_s, \\ q_i &= A_0 \equiv 0, && i = m_s + 1, \cdots, n, \end{aligned} \tag{17}$$

where we have

$$\begin{aligned} &0 < A_1 < A_2 < \cdots < A_s, \\ &1 \leqq m_1 < \cdots < m_s \leqq n, \\ &1 < s \leqq n. \end{aligned} \tag{18}$$

We now need only show that A_s and A_{s-1} are arbitrary within the range prescribed by (16). We may best show this by eliminating e between equations (12) and (15) and solving for C and c. For this purpose we write

$$e = (C - A_{s-2})R, \qquad R \equiv (y_{m_2} - y_{m_1})/y_{m_1}.$$

Setting this expression for e into (15) and solving for C and c, we obtain

$$\begin{aligned} C &= A_{s-1} + (A_s - A_{s-1})(1 + R), \\ c &= (C - A_{s-1})/(C - A_{s-2}). \end{aligned}$$

From (2) we see that R is positive, and from (6), that it is less than one.

Thus we see that, when A_s and A_{s-1} are arbitrary positive quantities satisfying (16), we will have

$$\begin{aligned} C &> A_{s-1} > A_{s-2}, \\ 0 &< c < 1; \end{aligned}$$

but C and c are arbitrary within these limits, so we see that inequalities (16) are the only restrictions imposed on A_s and A_{s-1}, and our theorem is proven.

For $m>2$, equations (2) and (3) are not as useful for the numerical calculation of g_i and y_m as are the equivalent equations

$$y_m^{m+1} - 2y_m^m + 1 = 0, \qquad m = 1, \cdots, n; \tag{19}$$

$$g_m = y_m - y_{m-1}, \qquad m = 1, \cdots, n, \tag{20}$$

where y_m is taken to be greater than one when m is greater than one.

The following table of values may prove useful in applying the theorem:

m	1	2	3	4	5	6	7
y_m	1.0000	1.6180	1.8393	1.9276	1.9651	1.9835	1.9919
g_m	1.0000	0.6180	0.2213	0.0883	0.0375	0.0185	0.0074

Since $y_m \doteq 2$ as $m \doteq \infty$, we may write

$$\sum_{8}^{\infty} g_i = 2 - y_7 = 0.0081,$$

and use as the upper bound simply

(e) $$z \leqq 0.0081\, q_8 + \sum_{1}^{7} g_i q_i.$$

To illustrate the use of the theorem and compare it with the bounds given by (*a*), (*b*), (*c*), and Lemma 3, we may apply each of them to the equation

(20) $$z^9 + z^8 - 512z^6 - 16z^5 + 16807z^4 + z^2 - 1 = 0.$$

The resulting bounds may be tabulated as follows:

(*a*)	(*b*)	(*c*)	(*d*)	Lemma 3
20	16	15	13	9.265

The direct use of Lemma 3 is, of course, not practical, since it involves the solution or approximate solution of an n-th degree equation.

THE PRODUCT OF A CIRCULANT MATRIX AND A SPECIAL DIAGONAL MATRIX

By UDO WEGNER, Darmstadt

In this MONTHLY, for May, 1932, vol. 39, p. 280, J. Williamson published a note, in which he proved by means of complicated determinantal relations that the matrix ZA, where A is the cyclic matrix

$$\begin{pmatrix} a_0 & a_1 \cdots a_{n-1} \\ a_{n-1} & a_0 \cdots a_{n-2} \\ \cdots & \cdots \\ a_1 & a_2 \cdots a_0 \end{pmatrix}$$

and Z the matrix

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \omega \cdots 0 \\ \cdots & \cdots \\ 0 & 0 \cdots \omega^{n-1} \end{pmatrix},$$

satisfies the equation

$$(ZA)^n - Det(A) \cdot E = 0,$$

where E is the unit matrix and ω a primitive nth root of unity. In what follows I shall show how this result can be obtained more simply.

Let B be written for the product ZA so that

$$B = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \omega a_{n-1} & \omega a_0 & \cdots & \omega a_{n-2} \\ \omega^2 a_{n-2} & \omega^2 a_{n-1} & \cdots & \omega^2 a_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ \omega^{n-1} a_1 & \omega^{n-1} a_2 & \cdots & \omega^{n-1} a_0 \end{pmatrix},$$

and let λ_0 be a latent root of B, i.e., Det $(B-\lambda_0 E) = |B-\lambda_0 E| = 0$. If the matrix $\omega(B-\lambda_0 E) = \omega B - \omega\lambda_0 E$ is transformed by the matrix

$$Q_1 = \left(\begin{array}{ccc|c} 0 & 0 \cdots & 0 & 1 \\ \hline 1 & 0 \cdots & 0 & 0 \\ 0 & 1 \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 \cdots & 1 & 0 \end{array}\right),$$

then

$$Q_1(\omega B - \omega\lambda_0 E)Q_1^{-1} = B - \omega\lambda_0 E,$$

since by the above transformation in ωB the first column is replaced by the last, the second by the first, the third by the second, and the first row by the last, the second by the first and so on. From the equation

$$0 = |B - \lambda_0 E| = \omega^n |B - \lambda_0 E| = |Q_1| \, |\omega B - \omega\lambda_0 E| \, |Q_1^{-1}| = |B - \omega\lambda_0 E|,$$

it follows that $\omega\lambda_0$ is a latent root of B.

If

$$Q_2 = Q_1^2 = \left(\begin{array}{cccc|cc} 0 & 0 & 0 \cdots & & 1 & 0 \\ 0 & 0 & 0 \cdots & & 0 & 1 \\ \hline 1 & 0 & 0 \cdots & & 0 & 0 \\ 0 & 1 & 0 \cdots & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & 1 & 0 & 0 \end{array}\right),$$

then

$$Q_2(\omega^2 B - \omega^2\lambda_0 E)Q_2^{-1} = B - \omega^2\lambda_0 E,$$

so that

$$0 = |B - \lambda_0 E| = \omega^{2n} |B - \lambda_0 E| = |Q_2| \, |\omega^2 B - \omega^2\lambda_0 E| \, |Q_2^{-1}| = |B - \omega^2\lambda_0 E|.$$

Accordingly $\omega^2\lambda_0$ is a latent root of B. Similarly it can be shown that $\omega^3\lambda_0$, $\omega^4\lambda_0$, $\cdots$, $\omega^{n-1}\lambda_0$ are latent roots of B. But these roots are all distinct and so $\lambda_0, \omega\lambda_0, \cdots, \omega^{n-1}\lambda_0$ are all the latent roots of B. Therefore the characteristic equation of B, $\phi(\lambda)=0$, has the form

$$\lambda^n - \lambda_0^n = 0.$$

Since the product of the latent roots of B is, apart from sign, the determinant of B or ZA,

$$\lambda_0^n = (-1)^{n-1}|Z||A| = (-1)^{n-1}\omega^{n(n-1)/2}|A| = |A|.$$

Therefore the characteristic equation of B is

$$\lambda^n - |A| = 0$$

and accordingly

$$(ZA)^n - |A|E = 0.$$

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION. PART III

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

The elder Āryabhaṭa has given rules for the extraction of the square and cube roots but not for finding the square and cube of a number. Brahmagupta has given rules for finding the cube as well as the cube root of a number but not for finding the square or the square root. An examination of these rules shows that they were meant to apply to numbers expressed in the modern place-value notation.

Let us first consider Brahmagupta's rule for finding the cube of a number. The rule is as follows:

Sthāpyaḥ+antya-ghanaḥ+antya-kṛtiḥ+
tri-guṇā+uttara-saṅguṇā ca *tat-prathamāt*
uttara-kṛtiḥ+antya-guṇā tri-guṇā ca+
uttara-ghanaḥ+ca ghanaḥ.[1]

The following is the prose rendering of this rule:

Antya-ghanaḥ (cube of antya) sthāpyaḥ (should be set down); antya-kṛtiḥ (square of antya) tri-guṇā (multiplied by three) uttara-saṅguṇā (multiplied by uttara) ca (and) tat-prathamāt (outside the first place of that) [sthāpyā (should be set down)], uttara-kṛtiḥ (square of uttara) antya-guṇā (multiplied by antya) tri-guṇā [tat-prathamāt sthāpyā]; uttara-ghanaḥ (cube of uttara) ca [tat-prathamāt sthāpyaḥ]; [evam (thus)] ghanaḥ (cube) [syāt (is)].

[1] *Brāhma-sphuta-siddhānta*, edited by Sudhākar Dvivedī, Ch. XII, verse 6. The plus sign + is used to indicate that the letters separated by it have been joined in the text by the rules of *sandhi* (conjunction of letters); the hyphen (-) indicates that the words separated by it have been joined in the text by the rules of *samāsa* (compounding of words).

Kṛitiḥ = square; tri = three; guṇā = saṅguṇā = multiplied; tat = that; prathama = first. In Sanskrit the conjunction *ca* is placed after the two words joined by it and not between them.

The reader should note that this rule gives the cube of a number of two digits. But the implication of the rule is that by repeated application of it the cube of any number can be found. The words *antya* and *uttara* are very significant. *Antya* means 'what exists at the end' and *uttara* means 'what comes after.' In the Brāhmī notation which preceded the modern place-value notation in India the number fifty-four was expressed as (50) 4. (50) cannot be called *antya*, as the principle of place-value was not used in the notation; 4 occurs at the end and must be called *antya*. As nothing occurs after 4, there is no *uttara*. Hence the rule is not applicable to the Brāhmī notation. We need not consider the Kharoṣṭhi notation as it disappeared from India more than a century before Brahmagupta. In the modern notation the number fifty-four is written as 54. Here the first place—the units' place is called the first place, *ekaṃ tu prathamaṃ sthānam*—is occupied[1] by 4 and the second or last place by 5. Hence 5 occurs at the end and is, therefore, *antya*; 4 comes after the digit 5, and so 4 is *uttara*. For *uttara* Mahāvīra uses the word *śeṣa* which means the same thing.

Far more significant than the words *antya* and *uttara* is the compound word *tat-prathamāt* italicised in the above rule. To find its significance let us compare the wordings of the above rule with those of the rules given by Mahāvīra (c. 850 A.D.) and Śrīdhara (b. 991 A.D.).

Mahāvīra's rule for cubing a number is:

Antyasya ghanaḥ kṛtiḥ + api
sā tri-hatā + *utsārya* śeṣa-guṇitā vā[2]
śeṣa-kṛtiḥ + tri + anta-hatā
sthāpyā + *utsārya* + evam + atra vidhiḥ.[3]
antyasya = of antya,
api = also;
sā = she (kṛtiḥ)
hatā = guṇitā
= multiplied.

Śrīdhara's rule for cubing a number is:

Sthāpyaḥ + antya-ghanaḥ + antya-kṛtiḥ
sthāna + *ādhikyaṃ* tri-pūrva-guṇitā ca
ādya-kṛtiḥ + antya-guṇitā tri-guṇā ca
ghanaḥ + tathā + ādyasya.[4]
tathā = similarly;
ādyasya = of ādya.

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[1] Hence Śrīdhara calls it *ādya* (what occurs at the beginning) or *pūrva* (what occupies the preceding place).

[2] Here *vā* means 'and'. This word has been previously used in this sense by Āryabhaṭa (*Daśa-gītikā*, verse 2) and Varāhamihira (Vṛhat Jātaka, XII, 19).

[3] *Gaṇita-sāra-saṃgraha* (edited by Rangācārya), Text, p. 15, verse 47.

[4] *Triśatikā* (edited by S. Dvivedī), p. 6, Rule 14.

In Mahāvīra's rule the last three words 'evam atra vidhiḥ' means 'such is the rule here.' This is expressed very tersely by the last word 'ghanaḥ' (cube, i.e., hence the cube of the number) in Brahmagupta's rule. Śrīdhara's rule is wanting in any such expression. In the rule Brahmagupta has used the word *uttara* thrice. Mahāvīra uses the word *śeṣa* twice and the rule is incomplete[1] as it indicates the operations corresponding to the three terms $a^3+3a^2b+3ab^2$. In a preceding rule he gives the correct formula $3a^2b+3ab^2+a^3+b^3$ for $(a+b)^3$. For *uttara* Śrīdhara uses the two synonyms[2] *pūrva* and *ādya*, the former being employed once and the latter twice. But for the word *sthānādhikyam*[3] Śrīdhara's rule could have been regarded as quite algebraical and applicable to any notation based on the additive principle and not on the principle of place-value.

If in Mahāvīra's rule we overlook the omission of the operation corresponding to the term b^3 in the expression $a^3+3a^2b+3ab^2+b^3$, the above three rules for finding the cube of a number differ only in the words italicised. The italicised word *sthānādhikyam* in Śrīdhara's rule means 'so as to produce increase in places.' The italicised word *utsārya* which occurs twice in Mahāvīra's rule means 'pushing one place to the right.' This very word has been used in this sense by the younger Āryabhaṭa in his rules for division and extraction of the square root,[4] by Śrīdhara in his rules for multiplication, squaring a number, and extraction of the square root, and by Bhāskara in his rules for multiplication and extraction of the square root. Hence the words *sthānādhikyam* and *utsārya* imply the same thing.

Mahāvīra's and Śrīdhara's rules may now be translated thus:

Set down the cube of *antya*; and then the product of the square of *antya* and three times the *śeṣa* (or *pūrva*) *after pushing it* (i.e. the product) *one place to the right or so as to produce increase in places*. Next set down the product of three times the *antya* and the square of *śeṣa* (or *ādya*) *after pushing it one place to the right* (or so as to produce increase in places). Then similarly set down the cube of *śeṣa* (or *ādya*). Such is the rule here.

I have purposely refrained from translating certain terms before finding their correct interpretations.

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[1] To complete the rule I would like to replace the words 'evam atra vidhiḥ' by the words 'atha śeṣa-ghanaḥ' (then the cube of *śeṣa*) which satisfy the metre equally well.

[2] See footnote 2 above.

[3] In the translation of *Triśatikā* published by Kaye in the *Bibliotheca Mathematica*, 3rd series, Vol. XIII, this word has been omitted. Śrīdhara has used the word *utsārya* five times as such and once in the past participle form *utsārita* (Rule 10). These words have also been omitted in the translation. As will be seen below, all these words indicate the use of the modern notation in India. The word *pada* meaning 'place' has been mistranslated wherever it occurs. Although the translation is said to be the work of Prof. Rāmānujāchāria M.A., the hand of Kaye is clearly visible behind these significant errors of omission and commission. I cannot think of an Indian scholar omitting or misinterpreting those very words which, when correctly interpreted, reflect great credit on his mother-land.

[4] In the latter rule the word occurs in the past participle form *utsārita*.

The question of increasing the number of places or of pushing one place to the right can arise only when the modern place-value notation is used. For example, let us find the cube of fifty-four.

In modern notation.	In any other notation.
$5^3 = 125$	$50^3 = 125000$
$3.5^2.4 = 300$	$3.50^2.4 = 30000$
$3.5.4^2 = 240$	$3.50.4^2 = 2400$
$4^3 = 64$	$4^3 = 64$
$54^3 = 157464$	$54^3 = 157464$

It will thus be seen that the words *antya* and *śeṣa* (or *purva* or *ādya*) cannot mean the component parts of a number. They must mean the digits used to express a number in our modern notation. Therefore *antya* means the digit occurring in the last place and *ādya* or *pūrva* the digit in the first place. Relative to *antya*, *śeṣa* (or *uttara*) is the digit which comes after *antya* and is, therefore, the digit in the first place. For, as stated before, only two places are contemplated in the rules as they stand.

The reader's attention is particularly invited to the remarkably close resemblance between the wordings of the rules of Brahmagupta and Śrīdhara. The word *sthānādhikyam* of the latter takes the place of the word *tat-prathamāt* of the former. This word indicates how the continued product of 3, the *uttara*, and the square of *antya*, should be set down relatively to the cube of *antya*. If these two numbers were to be added as usual, the word *tat-prathamāt* would have been superfluous. But Indian framers of *sūtras* or rules, far from indulging in the use of superfluous expressions, are brief to a fault.[1] The fifth-case ending in the word *tat-prathamāt* is also very significant. It implies removal or motion from the thing denoted by the base-word to which it is affixed. The word *tat-prathamāt* is connected with the word *sthāpyā* understood as the latter word has already been used once (though in the masculine form). These two words together mean "should be set down being removed from the first place of that (i.e., the preceding number)." Hence, Brahmagupta's *tat-prathamāt*, Mahāvīra's *utsārya*, and Śrīdhara's *sthānādhikyam* imply the same thing. The three rules quoted above are, therefore, merely three different statements of the same rule. It, therefore, follows that, like Mahāvīra's and Śrīdhara's rules, Brahmagupta's rule also is not applicable to any other notation than the modern place-value notation.

Let us now examine the rules given by the elder Āryabhaṭa and Brahmagupta for the extraction of the square and cube roots. These rules are:

(i) For the extraction of the square root.

Āryabhaṭa:

Bhāgaṃ haret+avargāt+nityaṃ
dvi-guṇena vargamūlena

[1] Vincent Smith, *The Oxford Students' History of India*, p. 36.

vargāt+varge śuddhe
labdhaṃ sthānātare mūlam.

(ii) For the extraction of the cube root.
Āryabhaṭa:
Aghanāt+bhajet+dvitīyāt
tri-guṇena ghanasya mūla-vargeṇa
vargaḥ+tri-pūrva-guṇitaḥ
śodhyaḥ prathamāt+ghanaḥ+ca ghanāt.
Brahmagupta:
Chedaḥ+aghanāt+dvitīyāt+
ghanamūla-kṛtiḥ+tri-saṅguṇā+āpta-kṛtiḥ
śodhyā tri-pūrva-guṇitā
prathamāt+ghanataḥ+ghanaḥ+mūlam.

Āryabhaṭa and Brahmagupta have given the same rule for the extraction of the cube root. The two statements have some important expressions in common. All these rules have been explained by A. N. Singh.[1] Āryabhaṭa's rules have been explained also by Rodet[2] and the present writer.[3] Singh and the present writer have adversely criticised the translation of Āryabhaṭa's above-mentioned rules published by Kaye in the *Journal of the Asiatic Society of Bengal*.[4] Further explanation of these rules is, therefore, unnecessary. I shall here try to show that these rules imply a knowledge of the place-value notation. Kaye writes that these rules are perfectly general (i.e., algebraical) and apply to all arithmetical notations.[5] If Kaye were right, how is it that the Greeks could not find the cube root of a number? Sir Thomas Heath writes that "in no extant Greek writer do we find any description of the operation of extracting the cube root."[6] Heron who has been assigned to the third century A.D. by this authority[7] has given a method of finding $4\frac{9}{14}$ as the cube root of 100.[8] But there is nothing to show that he could find the cube root of any number in general. On taking a very liberal view of his method the utmost credit that we can give him is that, when the integral part in the cube root of a number is known, he can find the fractional part approximately.

Again, the above rules occur in the midst of other rules relating to arithmetic or mensuration but not to algebra. The authors of these rules have omitted many essential topics and cannot be accused of incorporating rules which they did not use. The above rules are, therefore, arithmetical and are meant to apply

[1] *Bulletin*, Calcutta Math. Society, Vol. XVIII, pp. 123–140.
[2] *Journal Asiatique*, Vol. XIII (1879), pp. 397, 405–408.
[3] *Journal*, Bihar & Orissa Research Society, March, 1926, pp. 78–82.
[4] July, 1907, p. 493; March, 1908, pp. 119 & 120.
[5] *Ibid.*
[6] *History of Greek Mathematics*, Vol. I, p. 63.
[7] *Ibid.*, Vol. II, p. 306.
[8] *Ibid.*, p. 341.

to the notation used by their authors. As the Greeks with all their ingenuity could not apply the corresponding algebraical formula to their "additive notation" which was similar to the Indian Brāhmī notation in principle, it is but reasonable to infer that the notation employed by Āryabhaṭa and Brahmagupta was based not on the additive principle but on the principle of local value. This view is further supported by the expression *labdhaṃ sthānātare mūlam* which means 'the quotient is the part or figure of the root in the next place.' Although Kaye knows quite well that *sthāna* means *place* and not *term*, he has conveniently translated the expression as 'the quotient is the root to the next term.'[1] In March, 1908, he replaces it by the meaningless sentence "The quotient in a place set apart is the root."

From an examination of the rules of Āryabhaṭa quoted above Rodet holds that Āryabhaṭa performed the operations indicated by them on numbers written in figures with the value of position and the zero.[2] As Rodet learns from a commentary of Bhāskara's *Līlāvatī* the meanings of the terms *vargāt*, *avargāt*, *ghanāt* and *aghanāt*, Kaye thinks him to be wrong.[3] But the commentator has interpreted the terms after the manner of the elder Āryabhaṭa, Mahāvīra, the younger Āryabhaṭa, Caturvedācārya, Śrīdhara, and Bhāskara.

The words *avarga* and *varga* occurring in the forms *avargāt* and *vargāt* in the elder Āryabhaṭa's rule for the extraction of the square root may refer to notational places or, as Kaye interprets, to the component parts of the number whose square root is required. But the occurrence of the word *sthānāntare* renders Kaye's interpretation inapplicable. In the verse giving his alphabetic notation the elder Āryabhaṭa uses these two words *avarga* and *varga* to mean places corresponding respectively to the non-square units 10, 10^3, 10^5, etc., and to the square units 1, 10^2, 10^4, etc. This is also the view of Fleet.[4]

Mahāvīra (850 A.D.) gives the same rule for the extraction of the cube root as Āryabhaṭa and Brahmagupta, but he states it in two different ways. In the first statement he very appropriately uses the terms *bhājya* and *śodhya* respectively for *dvitīya aghana* and *prathama* (*aghana*). In his own statement of the same rule the younger Āryabhaṭa (c. 950 A.D.) clearly explains the terms of Mahāvīra. He writes that notational places (*padāni*) have respectively *ghana*, *bhājya* and *śodhya* for their names.[5] Ancient Indian mathematicians use the term *pada* in three different senses, (i) a notational place, (ii) the square root, (iii) the number of terms in a series. The second and third meanings are evidently inapplicable. In the immediately preceding rule which the younger Āryabhaṭa gives for the extraction of the square root he similarly names the places (*sthāna*) of a number as *odd* and *even*.

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[1] J.A.S.B., July, 1907, p. 493.

[2] *Journal Asiatique*, XIII (1879), p. 408.

[3] J.A.S.B., July, 1907, p. 494.

[4] J.R.A.S., 1911, p. 115 and p. 116, footnote 1.

[5] *Mahāsiddhānta*, Dvivedī's edition, XV, 8.

Caturvedācārya (Pṛthudakasvāmī) who wrote his commentary on Brahmagupta's *Brāhma-sphuṭa-siddhānta* in the latter half of the tenth century A.D. begins the explanation of Brahmagupta's rule for the extraction of the cube root as follows:

"Here (*i.e.*, in this rule) the first place of the number whose cube root is required is called *ghana* and the next two places towards the left are called *aghana*; then again the next one (place) is called *ghana* and the next two (places) are called *aghana*; and so until all the places are exhausted."[1]

This is exactly what Mahāvira means when he says "Ghanamekaṃ aghane dve" (the first place is *ghana* and the next two are—*aghana*) in his second statement of the rule for the extraction of the cube root. Śrīdhara also distinctly says this when he writes "ghanapadaṃ aghanapade dve ghanapadataḥ apāsya ghanam etc" (for one *ghana* (cubic) place there are two *aghana* (non-cubic) places; subtracting the cube from the cubic place etc.).

It will appear from the above that the terms *dvitīya* (second) *aghana* and *prathama* (first) *aghana* respectively mean 'second non-cubic place' and 'first non-cubic place.' Whatever be the notation, one must begin with the largest units to extract the square or cube root of an arithmetical number. For the extraction of the cube root the largest *ghana* (cubic) element must, therefore, be taken first. According to the rules given by Āryabhaṭa and Brahmagupta its place is followed on the right by the second *aghana* (non-cubic) place which is followed by the first *aghana* place. Then comes the *ghana* (cubic) place of the next lower order. This is quite in agreement with the classification of places explained by Caturvedācārya. Hence Rodet is not wrong in holding that the elder Āryabhaṭa's rules for the extraction of the square and cube roots imply the use of the modern notation.

Brahmagupta's rule for the extraction of the cube root being verbally almost the same as that of Āryabhaṭa, remarks made on the one apply equally to the other. It has already been shown that Brahmagupta's rule for cubing is applicable only to the modern notation. It would be unnatural if his rule for the extraction of the cube root, which is an inverse operation to that of cubing, were not likewise applicable only to the modern place-value arithmetical notation.

A GRAPHICAL SOLUTION OF THE QUARTIC

By J. D. GRANT,[2] University of Illinois

Some years ago Professor Running showed[3] that the real roots of a cubic or quadratic could be found from graphs made up solely of straight lines. These lines were all tangents to the discriminant of the equation studied so that the

[1] *Brāhma-sphuṭa- siddhānta*, Dvivedī's edition, p. 175.

[2] Died July 9, 1932.

[3] This MONTHLY, Vol. 28, 1921, page 415, also text *Graphical Mathematics*, 1927, p. 34.

method is an application of the tangential or line equations of Projective Geometry. The purpose of this note is to show that the same method may be applied to the quartic. The general reduced quartic may be written

$$z^4 + az^2 + bz + c = 0.$$

If $a \neq 0$, one may divide by a^2 and write $z/|a|^{1/2} = m$; and if $a = 0$, simply write $z = m$. Then according as a is positive, zero, or negative,[1] the equation takes one of the three forms

(1) $m^4 + m^2 + xm - y = 0$, where $x = b/a^{3/2}$, $y = -c/a^2$,

(2) $m^4 + xm - y = 0$, where $x = b$, $y = -c$,

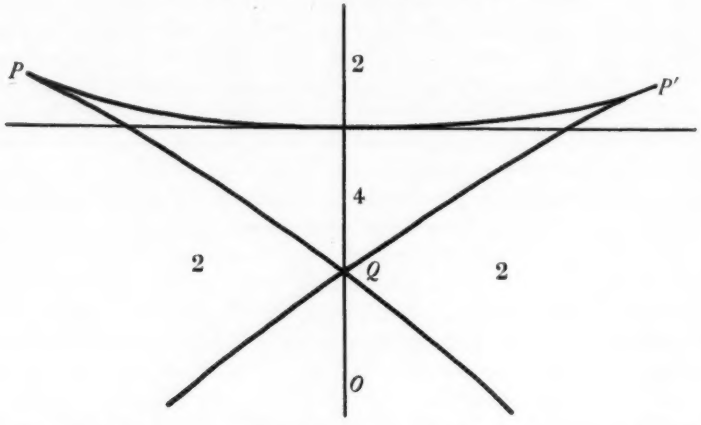
(3) $m^4 - m^2 + xm - y = 0$, where $x = b/(-a)^{3/2}$, $y = -c/a^2$.

It is clear that if the a, b, c are given real the resulting x and y must be real. The discriminant curves to which these families of lines are tangent are as follows:

(1) $27x^4 + 4(36y + 1)x^2 + 16y(4y + 1)^2 = 0$

(2) $27x^4 + 256y^3 = 0$

(3) $27x^4 - 4(36y + 1)x^2 + 16y(4y + 1)^2 = 0.$



Since $a < 0$ is a necessary condition for the existence of four real roots,[2] the third one of these curves is of particular interest. It is shown in the accompanying figure. The cusps, P, P', are at the points $(\pm 2\sqrt{6}/9, 1/12)$, and the node Q at $(0, -1/4)$. The curve divides the plane into three regions from the points of which 0, 2, or 4 tangents to the curve may be drawn as indicated in the figure.

[1] This is the method used by d'Ocagne in the study of the Quintic in *Traité de Nomographie*, 1899, page 341.

[2] Burnside and Panton, *Theory of Equations*, 1928, Vol. I, page 145.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

JACOBIANS OF THE SEMINVARIANTS S_i OF A BINARY P-IC

H. S. THURSTON, University of Alabama

A homogeneous isobaric polynomial in the coefficients of the binary form

$$f(x, y) \equiv \sum_{i=0}^{p} \binom{p}{i} a_i x^{p-i} y^i$$

is called a seminvariant of $f(x, y)$ if it is an invariant of $f(x, y)$ with respect to all transformations of the type

$$T_k\colon\ x = \xi + k\eta,\ y = \eta.$$

There exists a set of seminvariants

$$S_i = (-1)^{i-1}(i-1)a_1{}^i + \sum_{j=2}^{i}(-1)^{i-j}\binom{i}{j} a_0{}^{j-1}a_1{}^{i-j}a_j, \tag{1}$$

where i may take on the values $2, 3, \cdots, p$, such that every seminvariant of $f(x, y)$ is the quotient[1] of a polynomial in the seminvariants $a_0, S_2, \cdots, S_p$ by a power of a_0.

If each S_i be considered as a function of the independent variables $a_0, a_1, \cdots, a_p$, and J_m denote the Jacobian of $S_2, \cdots, S_{m+1}$ with respect to $a_0, a_1, \cdots, a_{m-1}$, it may be readily verified that

$$J_2 = a_0(4a_1a_3 - 3a_2{}^2)$$

$$J_3 = a_0{}^4(9a_2a_4 - 8a_3{}^2)$$

$$J_4 = a_0{}^8(16a_3a_5 - 15a_4{}^2)$$

$$J_5 = a_0{}^{13}(25a_4a_6 - 24a_5{}^2).$$

These results naturally suggest the general expression

$$J_m = a_0{}^{(m^2+m-4)/2}\left\{m^2a_{m-1}a_{m+1} - (m^2-1)a_m{}^2\right\}. \tag{2}$$

It is the purpose of this note to prove (2) for $m = 2, 3, \cdots, p-1$. Incidentally, for $m = 1$, this formula yields a_2, in accordance with the natural definition of J_1 as $\partial S_2/\partial a_0$.

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[1] Dickson, *Modern Algebraic Theories*, pp. 14–16.

From (1), we have

$$\partial S_i/\partial a_0 = \sum_{j=2}^{i}(-1)^{i-j}\binom{i}{j}(j-1)a_0^{i-2}a_1^{i-j}a_j$$

$$\partial S_i/\partial a_1 = (-1)^{i-1}i(i-1)a_1^{i-1} + \sum_{j=2}^{i-1}(-1)^{i-j}(i-j)\binom{i}{j}a_0^{i-1}a_1^{i-j-1}a_j$$

$$\partial S_i/\partial a_j = (-1)^{i-j}\binom{i}{j}a_0^{i-1}a_1^{i-j}, \text{ if } i \geqq j > 1$$

$$= 0, \qquad \text{if } i < j.$$

On account of the simpler notation involved we shall consider the Jacobian J_{m-1}, and later obtain (2) by replacing m by $m+1$.

$$J_{m-1} = \begin{vmatrix} \dfrac{\partial S_2}{\partial a_0} & \dfrac{\partial S_2}{\partial a_1} & a_0 & 0 & \cdots 0 \\ \dfrac{\partial S_3}{\partial a_0} & \dfrac{\partial S_3}{\partial a_1} & -3a_0a_1 & a_0^2 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial S_i}{\partial a_0} & \dfrac{\partial S_i}{\partial a_1} & (-1)^i\binom{i}{2}a_0a_1^{i-2} & (-1)^{i-1}\binom{i}{3}a_0^2a_1^{i-3} & \cdots \dfrac{\partial S_i}{\partial a_{m-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial S_m}{\partial a_0} & \dfrac{\partial S_m}{\partial a_1} & (-1)^m\binom{m}{2}a_0a_1^{m-2} & (-1)^{m-1}\binom{m}{3}a_0^2a_1^{m-3} & \cdots \binom{m}{2}a_0^{m-3}a_1^2 \end{vmatrix}$$

Removing the factors $a_0, a_0^2, \cdots, a_0^{m-3}$, from the third, fourth, $\cdots$, $(m-1)$ th columns respectively, and subtracting from the elements of the first and second columns respectively, the products of the elements of the third column by a_2 and $-2a_1$, we obtain a determinant in which all except one of the elements of the first row are zero. Having reduced this to a determinant of order $m-2$, and having removed a_0 from the first and second columns, we subtract from the elements of the first and second columns respectively, the products of those of the third column by $2a_3$ and $-3a_2$. The order may now be reduced to $m-3$, and a_0 factored from the first and second columns. Subtracting from the elements of these columns the products of the elements of the third column by $3a_4$ and $-4a_3$, the order may again be reduced. In general, we may reduce a determinant of order $m-k$ to one of order $m-k-1$ by subtracting from the elements of the first and second columns, the products of those of the third column by ka_{k+1} and $-(k+1)a_k$ respectively. Proceeding in this way, we finally obtain

$$J_{m-1} = a_0^{(m^2-m-6)/2} \begin{vmatrix} (m-2)a_{m-1} & -(m-1)a_{m-2} \\ (m-1)a_0a_m - m(m-2)a_1a_{m-1} & -ma_0a_{m-1} + m(m+1)a_1a_{m-2} \end{vmatrix}$$

which on expansion yields

$$a_0^{(m^2-m-4)/2}\{(m-1)^2a_ma_{m-2} - m(m-2)a_{m-1}^2\}.$$

Replacing m by $m+1$, we obtain (2).

AN UPPER LIMIT FOR THE ROOTS OF AN EQUATION

By R. J. MYERS, Bethlehem, Pa.

The following generalization of a theorem found in Dickson's Theory of Equations[1] is believed to be new.

Theorem: If in an equation having real coefficients

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

the first negative term is preceded by k coefficients which are either positive or zero, and if G denotes the greatest of the numerical values of the negative coefficients, then each real root is equal to or less than $1+(G/a_p)^{1/(k-p)}$, where a_p $(\neq 0)$ is the coefficient of the $(p+1)$th term, and p is less than k.

Proof: For positive values of x, $f(x)$ will be diminished in value or remain unchanged if we omit the terms a_0x^n, a_1x^{n-1}, $\cdots$, $a_{p-1}x^{n-p+1}$, $a_{p+1}x^{n-p-1}$, $\cdots$, $a_{k-1}x^{n-k+1}$, all of which are positive or zero, and if we change each coefficient from a_k to a_n inclusive from its present value to that of the negative coefficient which has the greatest numerical value (i.e. to $-G$). Then,

$$\begin{aligned} f(x) &\geqq a_px^{n-p} - G(x^{n-k} + x^{n-k-1} + \cdots + x + 1), \\ &\geqq \{a_p(x-1)x^{n-p} - Gx^{n-k+1} + G\}/(x-1). \end{aligned}$$

Dropping G from the right side of the equation does not affect the inequality if we assume $x>1$.

Hence

$$f(x) \geqq \frac{x^{n-p}[a_p(x-1) - Gx^{p-k+1}]}{x-1};$$

or, since $p-k+1 \leqq 0$, and $x>1$,

$$f(x) \geqq \frac{x^{n-p}[a_p(x-1) - G(x-1)^{p-k+1}]}{x-1}.$$

If now $x>1+(G/a_p)^{1/(k-p)}>1$, we have $(x-1)^{k-p}>G/a_p$, $(x-1)^{p-k}<a_p/G$, and $a_p(x-1)-G(x-1)^{p-k+1}>0$; and hence $f(x)>0$. Thus no value of x greater than $1+(G/a_p)^{1/k-p}$ can be a root.

Dickson gives two methods for the determination of the upper limit to the roots of an equation. The first is a special case of my theorem corresponding

[1] Dickson, First Course in the Theory of Equations, pp. 21–22.

to $p=0$. Moreover in all cases where p equals k the second method in Dickson is superior to mine, so my theorem can only be of use when there are at least three positive terms preceding the first negative term.

Consider the equation

$$x^4 + 18x^3 + x^2 - 36x - 14 = 0$$

and let us investigate the upper limit to the roots determined by these three methods.

By Dickson's first method the upper limit is found to be 4.31,
by his second method the upper limit is 2.80, and
by the method of this paper the upper limit is 2.41.
The only positive root is 1.50.

On Polygons and Circles

J. M. Feld, Brooklyn College

In a recent note[1] by J. R. Musselman on a problem proposed by W. H. Echols, Professor Musselman presented some elegant demonstrations of theorems on sets of equilateral triangles. Similar theorems can be established for polygons by a generalization of Professor Musselman's method.

Let $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ be two directly similar polygons, A_i and B_i being homologous vertices. If A_i and B_i are represented by the complex numbers a_i and b_i respectively, there obtain the n relationships

$$a_i = \lambda b_i + \mu \qquad (i = 1, 2, \cdots, n) \tag{1}$$

where λ and μ are complex. Therefore all determinants of the third order in the matrix

$$M = \begin{pmatrix} 1 & 1 \cdots 1 \\ a_1 & a_2 \cdots a_n \\ b_1 & b_2 \cdots b_n \end{pmatrix} = \begin{pmatrix} 1 \\ a_i \\ b_i \end{pmatrix}$$

vanish. Conversely if all the third order determinants of M vanish, the a_i and b_i are related as in (1) and the polygons A and B are directly similar. Hereinafter we write $M=0$ to indicate that all third order determinants of the matrix M vanish.

Let the k polygons $A_1A_2 \cdots A_n$, $B_1B_2 \cdots B_n$, $\cdots$ and $P_1P_2 \cdots P_n$ all be directly similar to $Q_1Q_2 \cdots Q_n$, the vertices bearing the same subscript being homologous. Furthermore let a_i, b_i, $p_i \cdots$ and q_i be complex numbers representing A_i, B_i, $P_i \cdots$ and Q_i respectively. Consequently we have the k conditions

$$\begin{pmatrix} 1 \\ a_i \\ q_i \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ b_i \\ q_i \end{pmatrix} = 0, \cdots, \begin{pmatrix} 1 \\ p_i \\ q_i \end{pmatrix} = 0. \tag{2}$$

---

[1] *On Equilateral Triangles.* This Monthly, vol. 39 (1932), p. 290.

Adding the k matrices appearing in (2) and dividing the resulting matrix by k we obtain

$$M_k = \begin{pmatrix} 1 \\ (a_i + b_i + \cdots + p_i)/k \\ q_i \end{pmatrix}.$$

Since by virtue of (2) every third-order determinant in M_k vanishes, $M_k = 0$. Therefore the centroids of the sets of homologous vertices of k directly similar n-gons are the vertices of another n-gon directly similar to the others.

If the n-gons are regular each may be regarded as the situs of a set of $t \leqq n$ distinct n-gons such that the homologous vertices of any two in the set can be brought into coincidence by rotating one about their common center. With a set of k regular n-gons it is therefore possible to associate numerous other regular n-gons by selecting sets of homologous vertices in various ways.

Let A_i^1 $(i = 1, 2, \cdots, n)$ be the vertices of a positive regular n-gon A^1. Similarly let $A_i^2, \cdots, A_i^k, \cdots, A_i^n$ be the vertices of the positive regular n-gons $A^2, \cdots, A^k, \cdots, A^n$ respectively.[1] The complex number representing point A_i^k is a_i^k. The regular polygons $A^1, \cdots, A^n$ are so placed that the n points A_1^i $(i = 1, 2, \cdots, n)$ are also vertices of a positive regular n-gon; and likewise the n points A_2^i are vertices of a positive regular n-gon. We therefore have

$$M^k = \begin{pmatrix} 1 & 1 & \cdots 1 \\ a_1^k & a_2^k & \cdots a_n^k \\ 1 & \omega & \cdots \omega^{n-1} \end{pmatrix} = 0, \qquad (k = 1, 2, \cdots, n) \tag{4}$$

and

$$M_k = \begin{pmatrix} 1 & 1 & \cdots 1 \\ a_k^1 & a_k^2 & \cdots a_k^n \\ 1 & \omega & \cdots \omega^{n-1} \end{pmatrix} = 0, \qquad (k = 1, 2) \tag{5}$$

where ω is a primitive nth root of unity.

Since the n-gons are regular we may cyclically permute the a_i^k in the second row of M^k without invalidating the set of equalities expressed by $M^k = 0$. Thus we subject the a_i^k in each matrix M^k to a cyclic permutation so that in all the resulting matrices the sum of superscript and subscript of the a in the jth column shall be congruent to $j \pmod n$. Therefore M^k becomes

$$M_*^k = \begin{pmatrix} 1 & 1 & \cdots 1 \\ a_{n-k+1}^k & a_{n-k+2}^k & \cdots a_{n-k}^k \\ 1 & \omega & \cdots \omega^{n-1} \end{pmatrix}.$$

Similarly we permute the a_1^i and the a_2^i in M_1 and M_2 producing

$$M_1^* = \begin{pmatrix} 1 & 1 & \cdots 1 \\ a_1^n & a_1^1 & \cdots a_1^{n-1} \\ 1 & \omega & \cdots \omega^{n-1} \end{pmatrix}, \quad M_2^* = \begin{pmatrix} 1 & 1 & \cdots 1 \\ a_2^{n-1} & a_2^n & \cdots a_2^{n-2} \\ 1 & \omega & \cdots \omega^{n-1} \end{pmatrix}$$

[1] The superscripts are not exponents except when used with ω.

and the relationships expressed in (4) and (5) are replaced by

$$M_*^k = 0, \quad (k = 1, 2, \cdots, n) \tag{4'}$$

and

$$M_k^* = 0, \quad (k = 1, 2) \tag{5'}$$

respectively.

Adding the matrices $M_*^k\,(k=1, 2, \cdots, n)$ and subtracting M_1^* and M_2^* from this sum we obtain a matrix M. Dividing M by $n-2$ we obtain

$$M^* = \begin{pmatrix} 1 & 1 \cdots 1 & \cdots 1 \\ b_1 & b_2 \cdots b_j & \cdots b_n \\ 1 & \omega \cdots \omega^{j-1} & \cdots \omega^{n-1} \end{pmatrix}$$

where[1]

$$b_j = \Big(\sum_{k=1}^{n} a^k{}_{n-k+j} - a_1^{n+j-1} - a_2^{n+j-2}\Big)/(n-2) \tag{6}$$

By virtue of (4') and (5') $M^*=0$. Therefore the points $B_j(j=1, 2, \cdots, n)$ represented by the complex numbers b_j are the vertices of a positive regular n-gon. Each vertex B_j is the centroid of a set of $n-2$ points selected in the manner indicated by (6). When $n=3$ this result reduces to Professor Musselman's theorem, viz., given two positively equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$ of any size or position in the plane, if we construct the positively equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$ then $A_3B_3C_3$ itself is a positively equilateral triangle.

A theorem on circles analogous to a previous one on polygons can be easily proved. A circle C is given by

$$z(t) = a + re^{i(\alpha+t)} (0 \leqq t < 2\pi)$$

where a (complex) represents its center, r (real), its radius and α (real), its phase angle.

Let the n circles C_k and their phase angles α_k be given by

$$z_k(t) = a_k + r_k e^{i(\alpha_k+t)}, \quad (k = 1, 2, \cdots, n).$$

The point $w(t_0) = (\sum_{k=1}^{n} z_k(t_0))/n$ is the centroid of the points on the n circles corresponding to $t=t_0$. As t runs from zero to 2π, $z_k(t)$ sweeps out circle C_k positively, starting from point $z_k(0)$, the phase angle of which is α_k, while $w(t)$ describes another positive circle $A+Re^{it}$ where

$$A = \Big(\sum_{k=1}^{n} a_k\Big)/n, \quad R = \Big(\sum_{k=1}^{n} r_k e^{i\alpha_k}\Big)/n.$$

The center of $w(t)$ is therefore the centroid of the centers of the n circles C_k, the

[1] In (6) subscripts and superscripts greater than n must be reduced by n.

radius of $w(t)$ is the modulus of R and the phase angle is the amplitude of R. We thus have the theorem: given n circles in the plane and given a point on each circle, let these points describe their respective circles positively at the same angular speed; then their centroid will describe a positive circle the center of which is the centroid of the centers of the given circles.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Linear Transformations in Hilbert Space and their Applications to Analysis. By Marshall H. Stone. American Mathematical Society Colloquium Publications, volume XV. New York, the American Mathematical Society, 1932. viii+622 pages.

Linguistic Analysis of Mathematics. By Arthur F. Bentley. Bloomington, Indiana, The Principia Press, 1932. xii+316 pages.

Symbolic Logic. By C. I. Lewis and C. H. Langford. New York, The Century Company, 1932. xii+506 pages. $5.00.

Reelle Funktionen. By Hans Hahn. Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 13. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1932. xii+416 pages. RM 28; bound, RM 30.

Physikalische Geodäsie. By F. Hopfner, Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 14. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1933. xii+434 pages.

Das Hauptproblem der Aeusseren Ballistik. By K. Popoff. Leipzig, Akad. Verlagsgesellschaft M.B.H., 1932. x+216 pages. 16.40 marks; bound, 18 marks.

Elementary Mathematics from a Higher Standpoint. By Felix Klein. Part I, Arithmetic, Algebra, Analysis. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, Macmillan, 1932. x+274 pages. $3.00.

Anschauliche Geometrie. By David Hilbert and S. Cohn-Vossen. Die Grundlagen der Mathematischen Wissenschaften, Band XXXVII. Berlin, Julius Springer, 1932. viii+310 pages.

An Attempted Proof of Fermat's Last Theorem by a New Method. By Correa M. Walsh. New York, G. E. Stechert, 1932. 42 pages. $1.00.

Elementary Differential Equations. By Lyman M. Kells, New York, McGraw-Hill Book Company, 1932. x+184 pages. $2.00.

Differential and Integral Calculus. By J. H. Neelley and J. I. Tracey. New York, Macmillan, 1932. x+496 pages. $4.00.

A Short Course in Trigonometry. By James G. Hardy. New York, Macmillan, 1932. x+182 pages; tables xxii+142 pages. $2.25.

The Principles of Financial and Statistical Mathematics. By Maximilian Philip. The College of the City of New York Series in Commerce, Civics and Technology. New York, Prentice-Hall, Inc., 1932. xx+406 pages.

REVIEWS

Studies in the Theory of Numbers. By Leonard E. Dickson. The University of Chicago Science Series, 1930. x+230 pages. $4.00.

This important volume has two claims to distinction: it contains an amazing number of new results in the theory of quadratic forms; and it represents a systematic treatment of the arithmetic theory of quadratic forms, starting from first principles. Either one of these accomplishments by itself would entitle the author and his students and collaborators (Arnold Ross, Gordon Pall, A. Oppenheim) to the lasting gratitude of all interested in the theory of numbers; the combination makes the book of quite outstanding value.

It would seem to call for some explanation why a systematic treatment *ab ovo* of an apparently well developed field such as the arithmetic theory of quadratic forms should be hailed as a noteworthy achievement. It will probably be to many readers, as it was to the reviewer, a shock to learn that in spite of the eminence of the mathematicians who have contributed to the theory (Gauss, Seeber, Smith, Zolatareff, Markoff, Frobenius, Minkowski, Eisenstein; to mention only some of those no longer living) and in spite of the very large number of textbooks on theory of numbers, we have no satisfactory exhaustive treatment of this field. In particular, the volumes of Bachmann "Die Arithmetik der Quadratischen Formen" are shown to be in important respects unreliable.

One can but admire the courage of an author who will undertake to rebuild the whole structure rather than to patch up the unsound portions. One can only guess at the amount of labor covered by the modest words of the preface; "It was no small task to write a satisfactory exposition." On the other hand, we know, in this country as well as in Europe, how much the theory of numbers owes to the insistence of Dickson on precision in the statement of theorems and to his uncanny ability to detect, and to mend, unsound arguments; it seems therefore only fair that to him and his students should belong the credit of writing the first reliable treatment of the arithmetic theory.

The work is divided into three parts containing eighteen chapters.

Part I. Pages 3–78. Arithmetic of ternary quadratic forms.

"The first six chapters deal with general aspects of quadratic forms, chiefly in three variables, with applications to quadratic Diophantine equations."

Chapter I. Quadratic forms in n variables.

Chapter II. Introduction to ternary quadratic forms; universal forms.

Chapter III. Representation of binary forms by ternary forms.

Chapter IV. Equivalence of indefinite, ternary, quadratic forms.

Chapter V. Genera and representations of numbers.

Chapter VI. Quadratic Diophantine equations.

Following a line of attack which he has inaugurated in other papers, Dickson investigates among other things in these chapters his "universal" forms, i.e., quadratic forms (with integral coefficients) which are capable of representing (for integral values of the variable) every integer, positive, negative, or zero. An outstanding new theorem states that "every universal, indefinite, ternary quadratic form is a zero form" that is, the number zero can be represented by values of the variables not all zero.

Of great elegance is the theorem: $f=ax^2+by^2+cz^2$, a, b, c integers $\neq 0$, is universal if and only if

(1). a, b, c not all of like sign.

(2). No two of a, b, c have a common odd factor >1.

(3). $abc\equiv 1 \bmod 2$ or $\equiv 2 \bmod 4$.

Chapter IV contains a correction or radical clarification of known results in Pell's equation, and a generalization of these results. Chapter V gives applications of the theory derived in Chapter IV to the representation of numbers by forms. Chapter VI contains the (first complete) proof that for every indefinite quadratic form f in five or more variables the equation $f=0$ has non-trivial solutions. Also, necessary and sufficient conditions for the existence of non-trivial solutions of $ax^2+by^2+cz^2+du^2=0$ are given, correcting and completing previous work by other authors; but still leaving, as is carefully stated, a desideratum (case when exactly two of a, b, c, d even, and $abcd/4\equiv 5 \bmod 8$).

Part II., pages 77–151, is entitled "Minima of indefinite quadratic forms," and contains:

Chapter VII. Minima of indefinite, binary, quadratic forms.

Chapter VIII. Minima of indefinite, ternary, quadratic forms.

Chapter IX. Minima of indefinite, quaternary, quadratic forms.

Chapter X. Tabulation of reduced, integral, ternary, quadratic forms which are indefinite but not zero forms.

This part centers around a theory originated by Markoff, extended by Frobenius and Remak. Here, again, it was necessary to reorganize the whole work to fill lacunae. Credit is given to G. Pall and A. Oppenheim for sharing the work of this portion. The following theorem may serve as a sample of the carefully polished form in which results are given:

Theorem: Any integral solution (a, b, c) of $x^2+y^2+z^2=3xyz$ with $c\geqq a\geqq 1$, $c\geqq b\geqq 1$, generates two new solutions (a, c, b'), $b'=3ac-b>c$; (b, c, a'), $a'=3bc-a>c$, which are distinct unless $a=b$. Repetition of this generating process, starting with the solution (1, 1, 1), will yield all sets of solutions in positive integers. The numbers forming a set are relatively prime in pairs.

The main theorems in this section are too technical to quote briefly.

Part III, pages 155–225, is headed "Miscellaneous investigations of quadratic forms," "The final three chapters are independent of each other and of the earlier chapters. They represent recent researches of especial interest concerning the geometrical reduction of positive forms, the determination of all universal zero forms, and the representation as sums of squares."

Chapter XI. Reduced, positive, quadratic forms. Their minima.

Chapter XII. Universal, zero, quadratic forms.

Chapter XIII. Number of representations as a sum of 5, 6, 7, or 8 squares.

In Chapter XI, Seeber's theory of reduced forms (supplemented and completed by Gauss, Eisenstein, Dirichlet) is given in simplified form, based on Gauss' lattice point representation of binary and ternary positive quadratic forms. For ternary forms, the proof of the reduction, and particularly the proof of uniqueness, is by its nature very disagreeable owing to the large number of possible cases which have to be examined separately.

Chapter XII gives the reduction to certain normal forms of the author's universal, indefinite, ternary or quaternary forms.

In the last Chapter, XIII, which is the only one to deal with the analytic theory of numbers in a broad sense, the author acknowledges the assistance of Gordon Pall and A. Oppenheim. It deals with the newer results of Mordell (and Hardy) on the number of representations of a positive integer n as the sum of s integral squares. This chapter will, to many readers, be particularly valuable because it is (as far as I know) the only place in the literature where this very abstract and delicate work is presented in connected fashion, with all necessary information (in very condensed form) concerning the ϑ-functions involved in the method, the Hardy function-theoretic tools and singular series, the Gaussian sums, etc. These twenty-seven pages impress the reviewer as a marvel of intelligent condensation.

A. J. KEMPNER

Nomographie. By M. Fréchet and H. Roullet. Paris, Armand Colin, 1928. 208 pages. Boards 12f; paper 10.50f.

The subject of Nomography is mainly a product of French inspiration and genius. It is not yet fifty years since d'Ocagne developed the idea of collinear points for the graphical representation of three variables in a plane; it is barely thirty since he published his most excellent Traité de Nomographie. The practical importance of this subject in engineering and in scientific experimental work is increasingly being realized—so it is peculiarly fitting and appropriate that we examine this work by a French mathematician and a French engineer.

The purpose of the book is to present to workers in the various fields of science the theory and methods involved in the construction of nomographic charts. Examples taken from physics, engineering, architecture, meteorology,

etc., serve to show the practical importance and use of the subject. Eighty figures illustrate the ideas and words of the authors. The mathematics involved can easily be understood by one who has had a good course in analytical geometry. Canonical forms which are necessary and sufficient to characterize the particular forms of nomograms are introduced, so that the investigator can readily find the type of diagram which suits his problem.

The book is divided into three parts. The first section of sixteen pages, which deals with the relationships between two variables, serves as an introduction. Here the authors present the fundamental notions of graphs and scales. The ordinary scale, the logarithmic and the general homographic scale are discussed in particular; by transformation of the scales other relationships between two variables can be represented by straight lines on the diagrams.

The second part of the book, one hundred and eight pages, treats the relationships between three variables. The development follows along historical lines; thus the first method presented is the cartesian coordinate chart with contour lines and the transformation of variables which permits these topographical curves to be represented by straight lines. The methods of Lalanne based on transforming one scale, and those of M. Lafay which transform both scales, are given and the correspondence between the original nomogram and its transform is discussed geometrically. We come now to the work of d'Ocagne and we are shown how to represent the relationship between three variables on three parallel lines; on two parallel lines and a curve; on two parallel lines and an oblique one—"abacs like an N"; on three concurrent lines; and finally on a circle with the third variable on a curve or a straight line. The section ends with a representation of the simultaneous solution of two relations between three variables.

The third section of forty-six pages deals with the relationships (a) between four variables when they can be put into certain simple canonical forms, and (b) between n variables when they can be expressed as $\sum f_i = f_n$ or $\prod f_i = f_n$ ($i = 1, 2, \cdots, n-1$). It concludes with a discussion of the advantages possessed by certain types of nomograms.

Fifty-five exercises scattered throughout the text and at the back of the book are given with hints as to the method of attack. These should prove valuable to the worker who wishes to construct a nomogram, as they cover all types of formulas. A study of some particular one will help him solve his individual problem. Certain paragraphs which deal with the geometric relationships between various types of nomograms are marked with an asterisk—to be omitted by a hurried reader; to the mathematician they will be of interest. It is not the aim of the book to compete with the volumes of d'Ocagne or Soreau; it is the aim of the authors to explain the subject of Nomography in simple terms and to show the non-mathematical worker in science how to use this tool. In this they have succeeded admirably.

J. R. MUSSELMAN

Handbook of Statistical Nomographs, Tables and Formulas. By J. W. Dunlap and A. K. Kurtz. New York, World Book Company, 1932. vii+163 pp. $6.00.

While nomograms have been in use in engineering and certain scientific fields, the worker in statistics has steered clear of graphic calculations. This is doubtless due to the fact that, as so much time must be spent in the numerical calculation of the values of the statistical measures before using a nomogram, the worker feels he might as well perform the last step also on his calculating machine. While Karl Pearson in his *Tables for Statisticians and Biometricians* (1914) includes two abacs or nomograms the present work with its twenty-eight charts is really a pioneer in the field. It will be interesting to see how much graphic methods of calculation will be stimulated by the handbook. The celluloid strip used for reading the nomograms can be improved; the lines on it should be decreased in thickness and the advertisement along the A scale removed, as its presence makes an exact reading difficult.

The list of 434 formulas collected from 48 books or articles on statistics has been written in a uniform notation and reference made to the books in which they can be found. In addition, errors in formulas occurring in these 48 books are noted. This portion of the handbook should prove most valuable. Statistics suffers from a lack of uniformity in notation and symbols—perhaps all writers should adopt this system as the standard. The reviewer regrets the inclusion of a formula like $\sigma_M = \sigma/(N-3)^{1/2}$ when $N<10$, for it must be entirely empirical as the theory of standard errors depends upon a normal curve which can hardly exist for so small a number of observations.

J. R. MUSSELMAN

Theorie der Raumkurven und krummen Flächen. By V. Kommerell and K. Kommerell. (Göschens Lehrbücherei. I. Gruppe: Reine und angewandte Mathematik, Band 20 und 21.) Berlin, de Gruyter, 1931. Band I. Krümmung der Raumkurven und Flächen. 205 pages. Rm. 10. Band II. Kurven auf Flächen. Spezielle Flächen. Theorie der Strahlensysteme. 194 pages. Rm. 10.

In presenting the fourth edition of their important work on the essentials of differential geometry, the authors have completely revised and rewritten the preceding edition of their three books which appeared as volumes 29, 44, and 62 of the Sammlung Schubert in 1921. The purpose and general plan of the earlier treatments remain unchanged and in the present edition the authors seem to have been especially successful in producing an elementary but scientifically sound introduction to the more extensive treatises of Bianchi, Eisenhart, Forsyth, and others. Numerous references to source material are given and as is stated in the preface, the diligent student of these volumes should be able to proceed to original investigation.

The method is that of classical differential geometry and the early introduction of the parametric representation of surfaces contributes greatly to the directness and simplicity of the treatment. Although vector methods are not

developed in the text, the geometric content of the result is frequently made more evident by the use of the vector notation. Naturally such portions of the text will be less easily read by the person who is not familiar with the elements of vector analysis. In keeping with the spirit of the times much greater emphasis has been placed on rigor in the present edition. Exact analytical definitions involving infinitesimals and limiting processes have replaced discussions of "infinitely near" or " consecutive" elements, these latter terms being retained only as modes of speech.

New subject matter mentioned in the preface includes the following: the parallel displacement of Levi-Civita; the Riemann-Christoffel curvature tensor with brief mention of its importance in the theory of relativity; the Gauss-Bonnet integral formula. The paragraphs dealing with the transformation of parameters and with the differential parameters of Beltrami have been completely recast.

Condensing the paragraph headings, we may say that a third of the first volume is devoted to the study of the space curve and its associated developable surfaces while the remainder of the volume deals with the curved surface. Here a rather complete study of the common properties of a curved surface in the neighborhood of an arbitrary point leads to a discussion of the mapping of one surface on another. The types of mapping considered are classified according as they preserve 1: angles (conformal mapping), 2: area, or 3: both angles and area (deformation). The last chapter of the volume considers geodesic curves and coordinates on a given surface, geodesic curvature and torsion of an arbitrary curve, and finally the formula of Gauss for the total curvature of a geodesic triangle on a surface.

The second volume opens with a discussion of the fundamental (Mainardi-Codazzi and Gauss) equations of a surface and the first chapter introduces differential parameters of the first and second orders (Beltrami) and applies them to the study of systems of curves on a surface and to the deformation of surfaces. Chapter two deals with the parallel displacement of Levi-Civita and extends the ordinary curvature tensor to the Riemann-Christoffel curvature tensor for an n-dimensional manifold. The next two chapters discuss surfaces of Weingarten and, in particular, the special cases—minimal surfaces and surfaces of constant curvature. The fifth chapter deals interestingly with ruled surfaces and triply orthogonal systems of surfaces. A very clear discussion of congruences of straight lines is given in the last forty pages of the text. In its essential points the treatment rests on the work of Kummer and several special types are considered.

The list of problems covering the text of the two volumes has been completely rewritten. The number has been more than doubled and they vary greatly in difficulty. Suggestions are frequently given and there are numerous references to source material.

The usefulness of these volumes as reference books is enhanced by a table of contents at the beginning of each volume and, at the end, by a list of authors

referred to and an index of subjects treated. The figures have been redrawn and their number increased. The books are attractively printed and the very few typographical errors noted will not cause confusion.

C. H. YEATON

Recent Developments in the Teaching of Geometry. By Jabir Shibli. Published by the author, State College, Pa., 1932. viii+252 pages.

This attractive volume is a worthwhile contribution to the literature of the teaching of mathematics chiefly because of its clarity and directness, as well as its timeliness. The first two chapters sketch the history of the teaching of geometry from classical antiquity to the year 1928 and provide an adequate background for the rest of the book. This material is the more welcome because of the inaccessibility of A. W. Stamper's "A History of the Teaching of Elementary Geometry," published in 1907, and long since out of print. The next three chapters, dealing with "Intuitive Geometry," "Introduction to Geometry," and "Foundations of Geometry," are rather good, although perhaps they leave a little to be desired because of their brevity. Chapters 6 and 7, which in a sense constitute the major portion of the book, present an excellent account of recent trends with regard to the treatment of book propositions and original exercises. The presentation of the aims and values of teaching demonstrative geometry is also good; the only objection would appear to be that in the discussion of transfer of training in connection with geometry the author makes no allusion to Mr. Betz's illuminating article on this subject in the fifth Yearbook of the National Council of Teachers of Mathematics.

On the whole, the book is well written, with an attractive typographical set-up, adequate footnote references, and a fairly adequate bibliography. It is unquestionably a valuable addition to the literature; taken together with the National Council's Fifth Yearbook we have a comprehensive survey of the field of demonstrative geometry teaching, and a substantial foundation upon which to base further reorganization and improvement of the 10th year mathematics of the future.

W. L. SCHAAF

Mathematics of Finance. Revised Edition. By H. L. Rietz, A. R. Crathorne, and J. C. Rietz. New York, Henry Holt and Company, 1932. xvi+346 pages.

Eleven years ago it was a pleasure to the writer to review the first edition of this book in the Journal of Commerce, New York City (Nov. 19, 1921), because there was so much to commend, in the formulation of the material, in the character and variety of the problems, in its teachableness as a class-room text, and in the workmanship of its manufacture.

The present text has been improved in the light of experience gained by teaching the previous edition to classes in colleges and universities throughout the country. And the authors, who are eminent teachers themselves, can say

that "Most of the changes are in the nature of simplifications." That is what we might expect to hear from living teachers, for art is simple. Some of the simplifications have become practicable through the prevalence of more extensive tables of compound interest and annuities. The tables are more than twice as extensive as in the first edition. The number of exercises has been greatly increased and their quality improved.

Originally, we were taken through Annuities Certain in two chapters covering 61 pages. Now, the same material is put into four chapters, 79 pages. The last two chapters are for reference, and treat Logarithms and Progressions, pages 252 to 282 inclusive.

C. C. GROVE

Philosophy and Modern Science. By H. T. Davis. Bloomington, Indiana, Principia Press, 1931. xiv+355 pages. Price $3.50.

The brief title is possibly misleading. The philosophy and modern science discussed are restricted to such as bear upon the physical sciences in a narrow sense. The work treats effectively historic problems but chiefly offers to the reader an introduction to the latest views in their historical and philosophical setting. The ground covered is fairly indicated by the successive chapter headings: Science—the modern metaphysics; is there an ether?; does matter drag the ether?; time, space and the fourth dimension; what is gravitation?; is matter finite?; the atom concept; the laws of chance; the metaphysics of the quantum theory and wave mechanics; the discovery and interpretation of cosmic radiation; what shall we believe?

The author is not only professor of mathematics, but also of the philosophy of natural science. It is in the latter capacity that he speaks in this book. A review commensurate in length with the excellence of this work would be inappropriate in a mathematical magazine. The style is brisk and is rendered lively by numerous brief quotations, being largely significant source material from the writings of the great discoverers whose results are under discussion, but in part in lighter vein, humorous or poetic. Lewis Carroll's immortal Alice speaks not infrequently. One catches hints of the enthusiasm of the original searchers after truth. In an undertaking with so ambitious an outlook (some 300 persons are mentioned by name) many great movements are necessarily dismissed with a few well chosen sentences, and elaborate evaluation yields to a condensed recital of facts and theories. Hence the book despite simple language, animated style and well sustained continuity, is not adapted to rapid reading.

There are here presented no original results of the author, either scientific or philosophical, but the choice of material is excellent, whether in classic problems already treated by a long line of distinguished writers or in the newest fields. The gathering of data, scientific, philosophical and literary is unusually effective. Technical notation is restricted to essentials, and the incompleteness

of current investigations is acknowledged by a frank undogmatic attitude of inquiry. The trained physicist may object to both matter and manner, the historian may balk at facts stated so baldly, the philosopher may miss his favorite topics, but the average intelligent reader who has not previously indulged in extensive specialized reading in the lines suggested, should find the book stimulating, informing, reliable, and should appreciate the abundance of explicit references giving him encouragement to extend his reading profitably. The proof-reading seems not on a par with the care devoted to other features in the production of this book.

A. A. BENNETT

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Lehigh University

The Chapter at Lehigh devoted the entire year to the study of the history of mathematics. Meetings were monthly with numerous papers by both faculty and student members. One open address was sponsored during the year. This was delivered by Dr. T. C. Fry, of Bell Telephone Laboratories, on March 17. Dr. Fry spoke on "Mathematics comes into its own." He gave a historical treatment of mathematical theories in the physical sciences. He believes that mathematics comes into its own when it successfully handles problems in physics.

At the April meeting the following officers were elected for the ensuing year. L. L. Smail, Director; Melvin Dresher, First Vice Director; J. W. Langhaar, Second Vice Director; W. C. Bachman, Secretary; J. A. Tempest, Treasurer; and G. M. Dewees, Librarian.

New members were publicly pledged in the Lehigh Chapel and initiated at a banquet at Hotel Bethlehem in May.

The Chapter reports a successful year.

TOMLINSON FORT, *Director*

LOCAL MATHEMATICS CLUBS

Undergraduate Mathematics Club of the University of Iowa

The officers for the year 1931–1932 were: Deane Montgomery, President; Marian Frey, Secretary-Treasurer; Dr. Allan T. Craig, Faculty Advisor. The officers are elected annually by the members of the club.

We have twenty-seven active members in our club.

The meetings and programs were as follows:

December 3, 1931: "Multiplication" by Professor R. P. Baker.
January 14, 1932: "Hyperbolic and elliptic functions" by Professor J. F. Reilly.
February 18, 1932: "Application of inversion to conics" by J. W. Querry.
March 17, 1932: "Curves associated with two given curves" by C. H. Fischer.
April 28, 1932: "Special conics" by E. G. Harrell.

MARIAN FREY, *Secretary*

The Mathematics Club of the Case School of Applied Science

The officers for 1931–1932 were: Ray Boyer, President; S. Eisler, Vice President; Edwin L. Smith, Secretary and Treasurer; Carl A. Cotman, Recording Secretary.

The purpose of the "Case Mathematics Club" is to promote an interest in the study of mathematics and also to show students the unity in the field of mathematics which can not be so effectively shown in the class rooms.

Any undergraduate or post-graduate student, as well as any member of the faculty is eligible for membership. Interest in mathematics is the sole qualification. Nominal dues of fifty cents ($.50) per term are required of active members. Our active members during the past year numbered twenty-five.

As usual, most of our meetings were held on the campus. Those not held on the campus were: The meeting of May 14, 1931 which was held at the Case Club; that of October 16, 1931 was held at Dean Focke's home; on April 29, 1932 we met at the Case Observatory; and on May 10, 1932, we met at the Case Club.

The meeting and programs were as follows:

May 14, 1931: "The unity of numbers" by Dr. Simon, Professor of Mathematics at Western Reserve University.
October 16, 1931: "The lure of mathematics" by Dr. Cairns, Professor of Mathematics at Oberlin College.
October 30, 1931: "Mathematics as an exact science" by Mr. Oldenburger, Instructor of Mathematics.
November 20, 1931: "Non-Euclidean geometry" by L. G. Henyey.
December 4, 1931: "The theory of relativity" by W. N. Simon, Jr.
December 18, 1931: "Theories of parallelism" by J. R. Parker.
January 16, 1932: "The theory of numbers" by L. P. Tarasov.
February 26, 1932: "Application of mathematics to the building and designing of air vehicles" by Dr. Hemke, Professor of Aeronautical Engineering.
March 11, 1932: "The inversion of the sphere" by Ray Boyer.
March 28, 1932: "Mathematics as a vehicle of thought" by Professor Thomas, Professor of Mathematics.
April 15, 1932: "Some theorems of projective geometry" by B. Napier.
April 29, 1932: "Experiences at Göttington" by Dean Focke, Professor of Mathematics.
May 10, 1932: "New fields of mathematics" by Dr. Clarke.

CARL A. COTMAN, *Secretary*

The White Mathematics Club of the University of Kentucky

The White Mathematics Club has been an organization on the University of Kentucky Campus for several years. It previously functioned under the auspices of the faculty of the Department of Mathematics. This year it was placed under the management of the student body.

The first meeting was held on October 15, 1931 under the leadership of Pi Mu Epsilon. At this meeting, the following officers were elected by popular vote: Dr. H. H. Downing, Advisor; Mary Allison Threlkeld, President; Elizabeth Ragland, Vice President; Anna Bruce Gordon, Secretary; Raymond Voll, Publicity Chairman.

The aim of the club is to increase the appreciation and knowledge of mathematics.

Students who are majoring or minoring in mathematics, those who have a definite interest in the subject, and graduate assistants in the department are eligible for membership. There are at the present time twenty active members in the club.

The meeting and programs were as follows:

October 15, 1931: "The problem of coloring maps" by Dr. Leon Cohen.

November 2, 1931: "The breaking up of numbers into factors" by Mary Allison Threlkeld.

December 3, 1931: "The foundations of mathematics" by T. L. Smith.

January 4, 1932: "The application of a problem in physics to mathematics" by J. H. Kirk.

February 11, 1932: "How our number system originated" by Nancy Duke Lewis.

March 10, 1932: "One to one correspondence" by Margaret LeStourgeon.

April 7, 1932: "A mathematical parody on Annabel Lee" by D. Ferrell.

April 22, 1932: The Mathematics Club entertained with a party, having as their guests all members of the faculty and their wives.

MARY ALLISON THRELKELD, *Secretary*

THE KAPPA MU EPSILON MATHEMATICAL FRATERNITY

This fraternity originated at Northeastern Teachers College in 1931 and has for its purpose the development of an appreciation of the beauty in mathematics, a greater interest in the subject and a broader understanding of it. Membership is extended to students who are majoring or minoring in mathematics and are in the upper half of their classes scholastically.

The chief emblems chosen by the fraternity are: The five pointed star and the pentagon. The pin and the key are pentagons. The seal is a star and the major part of the crest is a star. The wild rose is the fraternity flower. The pink of the wild rose and the silver of the star are the fraternity colors. In making the crest, five emblems, each representative of one of the sciences using mathematics, were chosen and placed around the star on the shield.

At the first Founder's Day banquet, held at Northeastern on April 18, 1931, the following officers were elected: Dr. Kathryn Wyant, Northeastern, President Pythagoras; Professor Ira S. Condit, Iowa State Teachers College, Vice President Euclid; Miss Lorene Davis, Secretary Diophantus; Professor L. P. Woods, Northeastern, Treasurer Newton; Miss Bethel DeLay, Historian Hypatia.

Chapters which have been installed since the founding are: Iowa Alpha, May 27, 1931, Iowa State Teachers College, Cedar Falls, Iowa; Kansas Alpha, May 20, 1932, Kansas State Teachers College, Pittsburg, Kansas; Missouri Alpha, Southwest Missouri Teachers College, Springfield; and Mississippi Alpha, May 30, 1932, Mississippi State Teachers College for Women, Columbus, Mississippi.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 16. *Proposed by G. A. Yanosik, New York University.*

Prove that the envelope of the circles whose diameters run from points on a parabola to its focus, is the straight line tangent to the parabola at its vertex.

E 17. *Proposed by Morgan Ward, California Institute of Technology.*

In Bierans de Haan's "Nouvelles Tables d'Intégrales Définies," Table 113, formula 1, page 162, it is stated that

$$-\int_0^1 \frac{\log x\,dx}{1+x+x^2} = \frac{2\pi^2}{27}$$

Show that this result is incorrect.

E 18. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Of all the right triangles whose areas exceed a million square units and whose three sides are integers without common factor, find that one whose perimeter is a minimum.

E 19. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

Find the radius of a sphere whose surface area and volume are each numerically equal to π times a four-place integer, and show that the solution is unique.

E 20. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The following letters represent the digits of a simple multiplication:

$$\begin{array}{cccccc} & & A & B & C & \\ & & & B & D & \\ \hline & C & E & E & B & \\ F & B & C & D & & \\ \hline F & G & C & G & B & \end{array}$$

Solve and show that the solution is unique.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3587. *Proposed by P. R. Rider, Washington University.*

The axes of two right circular cylinders of indefinite length intersect perpendicularly. Find the volume common to the two cylinders. (This problem actually arose in designing a tank car.)

3588. *Proposed by N. H. McCoy, Smith College.*

Find an explicit expression for f_n if $f_0=1$ and

$$f_{n+1} = (x^2 + y^2)f_n + 2ty\frac{\partial f_n}{\partial x} + t^2\frac{\partial^2 f_n}{\partial x^2},$$

where t is a parameter. Show that

$$(f_n)_{x=y=0} = K_n t^n, \quad K_n = \left.\frac{d^n(\sec 2t)^{1/2}}{dt^n}\right|_{t=0}.$$

3589. *Proposed by R. E. Gaines, University of Richmond.*

If a tangent to the cardioid $\rho=a(1+\cos\theta)$ at the point P_1 cuts the curve again in P_2 and P_3, the area of the segment cut off by the chord P_2P_3 is $\frac{3}{4}a^2(\phi-\sin\phi)$, where ϕ is the angle which the chord subtends at the origin.

3590. *Proposed by G. E. Raynor, Lehigh University.*

Prove that Simpson's (one-third) rule gives the correct value of the integrals

$$\int_0^\pi \sin^{2m}\theta d\theta, \quad \int_0^\pi \cos^{2m}\theta d\theta,$$

where m is a positive integer, if the number of intervals used is greater than two.

3591. *Proposed by B. F. Kimball, Schenectady, N. Y.*

Let the nth difference of $\log x$, with difference interval one, be denoted by $\Delta^n \log x$. Show that

$$\lim_{n\to\infty} n^x \log n\Delta^n \log x = \Gamma(x).$$

3592. *Proposed by V. Thébault, Le Mans, France.*

Given the tetrahedron $ABCD$; construct five equal spheres so that one of them shall be tangent to the remaining four, and each of these shall touch three faces of the tetrahedron.

3593. *Proposed by J. B. Reynolds, Lehigh University.*

Find the locus of the center of gravity of the volume cut from a homogeneous cube by a plane cutting off a constant volume.

SOLUTIONS

3501. [1931, 341]. *Proposed by E. B. Seitz, from "The Mathematical Visitor."*

Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken anywhere within the circle, find the average area of the triangle formed by joining the three points.

Solution by F. L. Wilmer, Odebolt, Iowa.

In this solution, which follows a method developed in W. Woolsey Johnson's Integral Calculus (1907), two triangles are considered different when and only when they are not identical. Let a be the radius of the given circle; let the given diameter be along Ox; let $B(r, \theta)$ be any point in the upper semi-circle; $C(\rho, \phi)$, any point in the lower semi-circle; and $A(R, \psi)$, any point in the circle. If N denotes the "number of cases" where A is restricted to the upper semi-cricle, then $N = (\pi a^2/2)^3 = \pi^3 a^6/8$.

We also have

$$\text{area } ABC = \tfrac{1}{2}[r\rho \sin(\phi - \theta) + R\rho \sin(\psi - \phi) + Rr \sin(\theta - \psi)],$$

and the expression on the right is positive if A lies in the sector of the circle with the angle θ. We shall restrict A in this manner, and this gives just half of the cases corresponding to N when A, B, C take all possible positions indicated above. Let M denote the desired average, then

$$M \cdot N = \int_{\pi}^{2\pi} \int_{0}^{\pi} \int_{0}^{\theta} \int_{0}^{\phi} \int_{0}^{a} [r\rho \sin(\phi - \theta) + R\rho \sin(\psi - \phi) + Rr \sin(\theta - \psi)] \, Rr\rho \, dR\,dr\,d\rho\,d\psi\,d\theta\,d\phi$$

$$= a^8[\pi^2 + 8]/18.$$

Hence

$$M = 4a^2[\pi^2 + 8]/9\pi^3.$$

3502. [1931, 341]. *Proposed by B. F. Finkel, Drury College.*

The centroid of a triangle is joined to its vertices and a point is taken at random in each of the three parts. Find the average area of the triangle formed by joining the three random points.

Solution by F. L. Wilmer, Odebolt, Iowa.

Let O be *any* point within the triangle ABC; and $OB=l$, $OC=m$, $OA=n$. In the triangles OBC, OCA, OAB let the angles opposite l and m, m and n, n and l, be, respectively, α_2 and α_1, β_2 and β_1, γ_2 and γ_1. Let A_1, B_1, C_1 be points within OBC, OCA, OAB, respectively. The area of OBC is $\frac{1}{2}l^2 \sin^2\alpha_1 (\cot \alpha_1 + \cot \alpha_2)$; and hence the "number of cases" is

$$N = \tfrac{1}{8}K^2(\cot \alpha_1 + \cot \alpha_2)(\cot \beta_1 + \cot \beta_2)(\cot \gamma_1 + \cot \gamma_2), \tag{1}$$

$$K = lmn \sin \alpha_1 \sin \beta_1 \sin \gamma_1.$$

Let the polar coordinates of A_1 with respect to OB be ρ_1, θ_1; those of B_1 with respect to OC, ρ_2, θ_2; those of C_1 with respect OA, ρ_3, θ_3.
Also

$$dN = \rho_1\rho_2\rho_3 d\rho_1 d\rho_2 d\rho_3 d\theta_1 d\theta_2 d\theta_3. \tag{2}$$

The area of $A_1B_1C_1$ is

$$\tfrac{1}{2}[\rho_1\rho_2 \sin (\alpha_1 + \alpha_2 + \theta_1 - \theta_2) + \rho_2\rho_3 \sin (\beta_1 + \beta_2 + \theta_2 - \theta_3) + \rho_3\rho_1 \sin (\gamma_1 + \gamma_2 + \theta_3 - \theta_1)]. \tag{3}$$

If M is the desired average, then $M \cdot N$ is equal to the sum of three integrals corresponding to the terms of (3), and from the symmetry of the expressions it will suffice to evaluate the first

$$I_1 = \tfrac{1}{2}\int_0^{k_3}\int_0^{k_2}\int_0^{k_1}\int_0^{h_3}\int_0^{h_2}\int_0^{h_1} \rho_1^2\rho_2^2\rho_3 \sin (\alpha_1 + \alpha_2 + \theta_1 - \theta_2) d\rho_1 d\rho_2 d\rho_3 d\theta_1 d\theta_2 d\theta_3; \tag{4}$$

$$h_1 = l \sin \alpha_1/\sin (\theta_1 + \alpha_1); \quad k_1 = \pi - \alpha_1 - \alpha_2;$$

$$h_2 = m \sin \beta_1/\sin (\theta_2 + \beta_1); \quad k_2 = \pi - \beta_1 - \beta_2;$$

$$h_3 = n \sin \gamma_1/\sin (\theta_3 + \gamma_1); \quad k_3 = \pi - \gamma_1 - \gamma_2.$$

The integrations with respect to ρ_1, ρ_2, ρ_3, θ_3 give easily

$$I_1 = \frac{1}{36} K^2 lm \sin \alpha_1 \sin \beta_1 (\cot \gamma_1 + \cot \gamma_2) \int_{\beta_1}^{\pi-\beta_2}\int_{\alpha_1}^{\pi-\alpha_2} \frac{\sin (C + x - y)}{\sin^3 x \sin^3 y}\, dxdy; \tag{5}$$

$$\theta_1 + \alpha_1 = x; \ \theta_2 + \beta_1 = y.$$

The final integration is also easy, and we obtain, after dividing the result by N, the first part M_1, of M.

$$M_1 = \frac{1}{18} lm \sin\alpha_1 \sin\beta_1 \left\{ 4\sin C + \sin C \frac{(\csc^2\alpha_1 - \csc^2\alpha_2)(\csc^2\beta_1 - \csc^2\beta_2)}{(\cot\alpha_1 + \cot\alpha_2)(\cot\beta_1 + \cot\beta_2)} \right.$$

(6)

$$\left. + 2\cos C \frac{\csc^2\beta_1 - \csc^2\beta_2}{\cot\beta_1 + \cot\beta_2} - 2\cos C \frac{\csc^2\alpha_1 - \csc^2\alpha_2}{\cot\alpha_1 + \cot\alpha_2} \right\}.$$

The remaining two parts, M_2 and M_3, of M are easily written by a cyclic interchange of the letters.

If ABC is equilateral with O at its centroid, the result is simple

$$M = 4^{-1}3^{-1/2}l^2 = 4^{-1}3^{-3/2}a^2, \tag{7}$$

where a is the length of the equal sides.

3525 [1932, 46]. *Proposed by H. A. Campbell, Omaha, Nebraska.*

A semicircle with center O has AC for the diameter at its base. The radius OC is prolonged to D so that $CD=OC=r$. The semi-circumference is bisected at Z, and on the arc ZC the point Y is taken so that ZY subtends at O an angle of 60°. Let the chord ZY produced cut OD in X; and with X as center and radius XD describe a semicircle on the same side of AD as that of the first one. Let B be any point on the arc AZ, and draw OB. Construct on this side of AD the angle DXM equal to the angle AOB, where M is the intersection of the side XM with the second semicircle. Let N be the projection of M on OD, and draw NB cutting again in T the semicircle at O. Prove or disprove that $NT=r$.

Editorial Note. If NT and r were equal, the angle AOB would be three times the angle ONB; and we know that no such construction will give this result for *any* position of B on the arc AZ. Determine the maximum error in this approximate method for trisecting an angle less than 90°, and show that the error is zero only for 90° and a zero angle.

Solution by W. B. Campbell, Rangoon, Burma

From the given construction we find that $OX=3^{1/2}r$, $XD=2r-OX$, $XN=XD\cos\theta$, where $\theta=\angle AOB$. Then $\tan\angle ONB=r\sin\theta/(r\cos\theta+OX+XN)$. Setting $\angle ONB=\theta/3+\epsilon$, we find

$$\epsilon = \tan^{-1}\left[\frac{\sin\theta}{3\cos\theta + 3^{1/2}(1-\cos\theta)}\right] - \theta/3. \tag{1}$$

Differentiating we find that ϵ is a maximum when

$$\cos\theta = \frac{3^{3/2}-1}{13}. \tag{2}$$

Since $\epsilon=0$ for $\theta=0°$ or 90°, it does not equal zero for any angle in the first quadrant, as the function has only one turning point in this interval.

From (2) the value of θ for the first quadrant is $\theta = 71°10'9''$, $\theta/3 = 23°43'23''$, $\angle ONB = 23°50'45''$, $\epsilon = 7'22''$.

A Note by Otto Dunkel. For very small angles the error is given approximately by $\epsilon = \theta^3/393$. The geometric fact stated in the first sentence of the editorial note on the problem can be utilized in a more interesting and successful manner. If $\angle ONB$ is a good first approximation to a third of $\angle AOB = \theta$, a better second approximation $\angle ON'B$ is obtained by finding N' on ON, or ON produced, so that $TN' = r$. If the first approximation is too large, the second is too small, and conversely. If BN' cuts the circle (O) in T', we find a third and still better approximation $\angle ON''B$ in the same manner, and so on. If the error in the initial approximation is ϵ, then the error in the second approximation is approximately $\epsilon' = -4\epsilon\theta^2/27$, if θ is small. Hence in this case the successive approximations converge rapidly. See the article by E. C. Kennedy, *Angle Division*, in this MONTHLY, vol. 39 (1932), p. 478.

Also solved by A. Pelletier and R. C. Staley. One unsigned solution was received from Carnegie Institute of Technology.

3535 [1932, 174]. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The pedal curve p with respect to a point P of a given curve c is defined as the locus of the feet of the normals from P to the tangents of c, and c is known as the negative pedal of p. Prove that the negative pedals of circles and straight lines with respect to a point P, not on the circles or straight lines, are respectively central conics and parabolas having a focus at P. The parabolas are tangent to their pedals and the central conics are doubly tangent to theirs.

Solution by Ethel I. Moody, Sweet Briar College

Let the point P be taken as the origin. The equation of any straight line not passing through P may be written in the form

$$Ax + By + 1 = 0. \tag{1}$$

The lines through P may be represented by the equation

$$x + \lambda y = 0. \tag{2}$$

Each line of the system (2) intersects the straight line (1) in a point the coordinates of which are

$$[-\lambda/(A\lambda - B),\ 1/(A\lambda - B)]. \tag{3}$$

Each tangent to the negative pedal of (1) with respect to P passes through a point (3) and has slope λ, and, therefore, this system of tangents can be represented by the equation

$$\lambda^2(Ax + 1) - \lambda(Bx + Ay) + By + 1 = 0. \tag{4}$$

The negative pedal of the line (1) with respect to P is the envelope of the system

(4), and its equation is

$$B^2x^2 - 2ABxy + A^2y^2 - 4Ax - 4By - 4 = 0. \tag{5}$$

The curve represented by equation (5) is a parabola having the origin as focus and the line $Ax+By+2=0$ as directrix. Since one of the lines of the system (2) is perpendicular to the line (1), this line itself satisfies the conditions for a tangent to the negative pedal and is therefore tangent to the parabola (5).

The equation of any circle not passing through the origin P may be written in the form

$$(x - \alpha)^2 + (y - \beta)^2 = r^2, \ (\alpha^2 + \beta^2 \neq r^2), \tag{6}$$

and the lines perpendicular to the lines of the system (2) may be represented by the equation

$$y = \lambda x + k. \tag{7}$$

To be tangent to the negative pedal of the circle (6) with respect to P, a line must belong to the system (7) and intersect the line of the system (2), determined by the same value of λ, in a point the coordinates of which satisfy equation (6). This system of tangent lines is represented by the equation

$$\lambda^2(x^2 - 2\alpha x + \alpha^2 + \beta^2 - r^2) + 2\lambda(\alpha y + \beta x - xy) + (y^2 - 2\beta y + \alpha^2 + \beta^2 - r^2) = 0. \tag{8}$$

The result of eliminating λ between equation (8) and its first partial with respect to λ is

$$\begin{aligned}[x^2 - 2\alpha x + \alpha^2 + \beta^2 - r^2][x^2(r^2 - \alpha^2) + y^2(r^2 - \beta^2) \\ - 2\alpha\beta xy + 2\alpha x(\alpha^2 + \beta^2 - r^2) \\ + 2\beta y(\alpha^2 + \beta^2 - r^2) - (\alpha^2 + \beta^2 - r^2)^2] = 0.\end{aligned} \tag{9}$$

The first factor of (9) equated to zero represents two lines of the system (8) which are parallel to the y-axis. The second factor of (9) equated to zero is the negative pedal of the circle (6) with respect to P, and, since $\alpha^2+\beta^2 \neq r^2$, this negative pedal is either an ellipse or a hyperbola according as $\alpha^2+\beta^2$ is less than or greater than r^2. The center of this conic is (α, β) and the foci are $(0, 0)$ and $(2\alpha, 2\beta)$. If $\alpha^2+\beta^2<r^2$, the curve is the locus of points the sum of whose distances from the foci is $2r$, and, if $\alpha^2+\beta^2>r^2$, the curve is the locus of points the difference of whose distances from the foci is $2r$. The principal diameter

$$\alpha y - \beta x = 0 \tag{10}$$

of the conic is also a diameter of the circle (6), and hence is perpendicular to the tangents to this circle at the points in which it intersects the circle. Since the line (10) passes through P, these tangents to the circle are also tangent to the negative pedal of (6) with respect to P. But tangents to a conic which are perpendicular to a principal diameter must have their points of contact on this

diameter, and therefore the circle (6) and its negative pedal with respect to P have common tangent lines at the two points in which the line (10) intersects the circle (6).

Also solved by E. M. Berry, W. B. Campbell, Marie M. Johnson, W. V. Parker, and C. A. Rupp.

A Note by Otto Dunkel. The geometry of this problem is rather simple. In the general case of any curve p let M, M' be two neighboring points of p, and let the perpendiculars to PM and PM' through M and M', respectively, meet in Q'. Since P, M, M', Q' lie on a circle with the diameter PQ', the limiting position Q of Q' on MQ', that is the point of tangency of MQ' with c, is easily found as follows: Let the perpendicular bisector of PM cut in C the normal at M to p, and produce PC to Q so that $CQ=PC=CM$.

If p is a straight line l, let K and K' be the feet of perpendiculars from P and Q to l; then $PK+QK'=2CM=2PC$. Produce PM to cut QK' extended in S. Then $QS=PQ$, since $K'S=PK$. Hence Q lies on a parabola with the focus P and with the directrix parallel to l so that K is the mid-point of the perpendicular to it from P. Obviously the parabola is tangent to l at K. The proof also shows that the tangent to the parabola at Q bisects the angle between the focal ray PQ and the line through Q parallel to the axis KP. We also see that the envelope of circles having the focal ray PQ as a diameter is the tangent at the vertex.

If p is a circle with center C_0 and radius r, the same construction gives Q, where in this case the normal at M is the radius C_0M. For convenience suppose that P is inside the circle p, and let PC_0 cut this circle in the ends of the diameter DD'. Let the parallel to MC_0 through Q cut DD' in P'; then $PC_0=C_0P'$, since $PC=CQ=CM$. Also $P'Q=2C_0C=2C_0M-2CM$, and $PQ=2CM$. Therefore $P'Q+PQ=2C_0M=2r$, and Q lies on the ellipse with foci P, P'. This ellipse is tangent to the circle at D and D'. This also shows that the envelope of circles with the focal ray PQ as diameter is the auxiliary circle p. The construction shows that the normal at Q to the ellipse is parallel to MP, and since MCP is isosceles, the normal bisects the angle between the focal rays PQ and $P'Q$. The case of P outside p is treated in a similar manner.

3537 [1932, 175]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters H, V, and P designate a system of mutually perpendicular coordinate planes, and the letters α, β and γ the angles which a plane Q makes with H, V, and P respectively. If none of these angles exceeds 90°, what are the extreme limits of the sum $\alpha+\beta+\gamma$?

Solution by E. M. Berry, Lynchburg College

Let H, V, and P be the zy, xz and xy planes respectively and let $x\cos\alpha+y\cos\beta+z\cos\gamma-p=0$ be the plane Q. Then α, β, and γ are the angles HQ, VQ, and PQ respectively.

Then we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1}$$

The sum $S=\alpha+\beta+\gamma$ is to be a maximum or a minimum. Substituting the value of α from (1) we have

$$S = \beta + \gamma + \cos^{-1}(1 - \cos^2\beta - \cos^2\gamma)^{1/2}. \tag{2}$$

The partial derivatives $\partial S/\partial\beta$ and $\partial S/\partial\gamma$ must equal 0. This gives us the two conditions

$$\cos\beta\sin\beta = \cos\gamma\sin\gamma = (1 - \cos^2\beta - \cos^2\lambda)^{1/2}(\cos^2\beta + \cos^2\gamma)^{1/2}. \tag{3}$$

From these we have $\sin 2\beta = \sin 2\gamma$.

$$\therefore \quad \beta = \gamma \quad \text{or} \quad \beta + \gamma = 90^\circ.$$

Substituting $\gamma=\beta$ in equation (3) we get

$$3\cos^2\beta = 1 \text{ or } \cos\beta = 0$$

$$\therefore \quad \alpha = \beta = \gamma = 54^\circ 44', \text{ and } S = 164^\circ 12', \text{ or}$$

$$\alpha = 0, \beta = \gamma = 90^\circ, \text{ and } S = 180^\circ.$$

By trying values near by, we find the former is the minimum and the latter is the maximum for the sum S.

If $\beta+\gamma=90^\circ$ we find from (1) that $\alpha=90^\circ$ and the sum S is 180°. This is the maximum for the sum.

If the angles were allowed to exceed 90° then 180° would not be a maximum value for the sum.

Also solved by W. B. Campbell, F. Underwood and the Proposer.

A Note by Otto Dunkel. The determination of the extremes may be put in a form which shows more clearly the variation of S. Let the unit sphere with its center at the origin cut the rectangular axes in A, B, C, and let P be the point in which the sphere is cut by a ray from the origin in the first octant. In terms of arcs of great circles we have $AP=\alpha$, $BP=\beta$, $CP=\gamma$. If P is at A, $\alpha=0$ and $\beta+\gamma=\pi$; if P is on the side BC of the spherical triangle ABC, $\alpha=\frac{1}{2}\pi$, $\beta+\gamma=\frac{1}{2}\pi$. For these positions of P, $S=\alpha+\beta+\gamma=\pi$. If P is at any point inside, then by geometry $\frac{1}{2}\pi<\beta+\gamma<\pi$. Consider the parametric curve $\beta+\gamma=t$ (constant), where $\frac{1}{2}\pi<t<\pi$, $0<\alpha<\frac{1}{2}\pi$. For this curve, we have

$$\frac{d\alpha}{d\beta} = \frac{2\cos t\sin(t - 2\beta)}{\sin 2\alpha}. \tag{1}$$

Since $\cos t<0$, $\sin 2\alpha>0$, we see that α decreases from $\pi-t$ as β increases from $t-\frac{1}{2}\pi$ until $\beta=\frac{1}{2}t$. After this point α increases until $\beta=\frac{1}{2}\pi$ and then $\alpha=\pi-t$. Hence the maximum for S on this curve is π at the two points where it cuts BA and CA. This maximum $S=\pi$ is attained therefore at each point of the perimeter and at no point of the interior.

Also the minimum value of S for this curve occurs at the point where it cuts the median arc AM_a, where M_a is the mid-point of side BC. We now examine the values of S along this median, where $\beta=\gamma$. We have

$$\frac{dS}{d\beta}=\frac{4\cos(\alpha+\beta)\sin(\alpha-\beta)}{\sin 2\alpha}. \tag{2}$$

As before $\frac{1}{2}\pi\leqq\alpha+\beta<\pi$, $\cos(\alpha+\beta)\leqq 0$, $\sin 2\alpha\geqq 0$. For $\beta=\frac{1}{4}\pi$, $\alpha=\frac{1}{2}\pi$; for $\beta=\frac{1}{2}\pi$, $\alpha=0$; and at these points only is the denominator in (2) zero. As β increases α decreases. Hence S decreases as β increases from $\frac{1}{4}\pi$ until $\beta=\alpha$; after this point S increases. Therefore there is a minimum S at just one point of the triangle, $\alpha=\beta=\gamma$, the pole of the triangle ABC. Very simple theorems of spherical geometry have been used above. The proof could be made entirely geometrical by assuming certain properties of the spherical ellipse $\beta+\gamma=t$.

3539 (1932, 175). *Proposed by R. E. Gaines, University of Richmond.*

A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

Solution by C. H. Fischer, University of Iowa

Since the rod is resting on the table it is obvious that the center must also lie on the table, i.e. within the given circle of radius r. The axes may be chosen so that the x-axis is parallel to the rod, and so that the rod then lies in the first two quadrants. The equation of the given circle (table) is $x^2+y^2=r^2$. If the rod is moved so that it remains parallel to the x-axis and its left end continues to make contact with the circumference of the given circle, the center of the rod describes an arc of a circle of radius r and center at $(a, 0)$. A similar procedure with the right end of the rod making contact with the circumference causes the center to describe an arc of a circle of radius r and center at $(-a, 0)$. These auxiliary circles intersect on the y-axis at $[0, (r^2-a^2)^{1/2}]$.

Since $r>a$, it is seen that if the center of the rod lies within the portion of the semicircle bounded by the two auxiliary circles, $(x-a)^2+y^2=r^2$, $(x+a)^2+y^2=r^2$, and the x-axis, neither end of the rod can project over the edge of the table.

If the center of the rod lies to the left of the auxiliary circle, $(x-a)^2+y^2=r^2$, the left end of the rod projects over the edge of the table. Similarly, if the center of the rod lies to the right of the auxiliary circle $(x+a)^2+y^2=r^2$, the right end of the rod projects beyond the edge of the table. Hence, in the region bounded by the given circle and the two auxiliary circles, i.e., where the center of the rod lies to the left of the first auxiliary circle and to the right of the second auxiliary circle, both ends of the rod project over the edge of the table. If the center of the rod lies in either of the two portions of the region, each half bounded by the three circles and the x-axis, then only one end of the rod will project over the edge of the table.

The probabilities may be obtained by dividing the areas of the above regions by the area of the semicircle, $\frac{1}{2}\pi r^2$. After calculating these areas in the usual manner we find:

(1). The probability that neither end of the rod will project over the edge of the table is

$$\frac{2}{\pi}\arccos\frac{a}{r} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2}.$$

(2). The probability that one end of the rod will project over the edge of the table is

$$\frac{4a}{\pi r^2}(r^2 - a^2)^{1/2} - \frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{4}{\pi}\arccos\frac{a}{r} + \frac{4}{\pi}\arccos\frac{a}{2r}.$$

(3) The probability that both ends of the rod will project over the edge of the table is

$$\frac{a}{\pi r^2}(4r^2 - a^2)^{1/2} - \frac{2a}{\pi r^2}(r^2 - a^2)^{1/2} + \frac{4}{\pi}\arcsin\frac{a}{2r} - \frac{2}{\pi}\arcsin\frac{a}{r}.$$

It is possible that another choice of independent variable might lead to a different result, in somewhat the same manner as the various results of Bertrand's paradox are obtained. Another quite different formulation of the problem which the writer has worked through leads, however, to the identical result above.

Also solved by Thomas Butler, V. F. Ivanoff, and W. M. Rust, Jr.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Dr. Niels Bohr, professor of physics at the University of Copenhagen, expects to visit the United States during the summer of 1933.

Assistant Professor W. L. Ayres has received the Henry Russel Award for 1932 at the University of Michigan. This prize is awarded annually to a young man of the faculty of that University for past achievement and future promise in research. The prize was established in 1925.

Dr. William Bowie, of the United States Coast and Geodetic Survey, has been elected a member of the State Russian Geographical Society.

Professor A. H. Compton, of the University of Chicago, has been elected a corresponding member of the Prussian Academy of Sciences.

Dr. Irving Langmuir, of the General Electric Company, has been elected a member of the Academy of Sciences of Halle.

Dr. Robert A. Milligan was presented with the distinguished service medal of the Roosevelt Memorial Association on October 27, 1932.

At the University of Chicago Professor H. E. Slaught has retired from active service in the department of mathematics with the title of Professor Emeritus, having been connected with the University since it opened its doors in the autumn of 1892. He was a Fellow in mathematics at the University of Chicago from 1892 to 1894, and was one of the first group of graduate students who received the Ph.D. degree in mathematics from the University. From 1894 on he has been a member of the staff of the department of mathematics. He was a central figure in the founding of the Mathematical Association of America, served a term as its president, and has been for many years a managing editor of this MONTHLY. He has also served the American Mathematical Society with distinction as secretary of the Chicago Section and in numerous other capacities. The inspiring significance of his influence in college and university mathematical instruction is widely recognized.

Professor Oswald Veblen, who, as previously announced, will join the staff of the Institute of Advanced Study at Princeton, has presented his resignation to the trustees of Princeton University. Professor Veblen has been connected with Princeton for 27 years.

Dr. O. E. Brown has been promoted to an assistant professorship at the Case School of Applied Science.

Dr. Laura Guggenbuhl of Hunter College has been promoted to an assistant professorship of mathematics.

Dr. H. L. Slobin, head of the department of mathematics at the University of New Hampshire, has been appointed dean of the Graduate School.

Mr. Paul A. Benitz and Miss Isabel C. McLaughlin have been appointed to instructorships at Hunter College.

Professor E. H. Moore, professor emeritus of the University of Chicago, died December 30, 1932, aged 70 years. Professor Moore was one of the outstanding mathematicians of the present generation. He began his teaching career as instructor at the Academy at Northwestern University in 1886. From 1887 to 1889 he held a position as tutor at Yale. From 1889 to 1892 he was associated with Northwestern University, first as assistant professor and later as associate professor. In 1892 he was made professor of mathematics at the University of Chicago, and he held this position until his retirement a few months ago. He was honored by various universities both in America and abroad, receiving honorary degrees from Yale, Wisconsin, and Clark Universities in the United States, and from Toronto University and the University of Göttingen. He served as vice-president of the American Mathematical Society, 1898–1900, and as president, 1900–1902. In addition to this, he served on the editorial staffs of various mathematical journals, both in the United States and abroad. He was interested in the founding of the Mathematical Association of America and was a charter member.

THE SEVENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The seventh annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Swarthmore College on Saturday, November 26, 1932, Professors Dresden and Kline presiding.

The attendance was seventy-four, including the following thirty-nine members of the Association: S. S. Cairns, P. A. Caris, J. W. Clawson, Mary L. Constable, E. S. Crawley, J. E. Davis, Arnold Dresden, Tomlinson Fort, Orrin Frink, Jr., J. S. Gold, Michael Goldberg, H. V. Gummere, J. R. Kline, P. A. Knedler, V. V. Latshaw, Marguerite Lehr, F. L. Manning, R. W. Marriott, D. L. McDonough, Edith McDougle, J. A. Miller, H. H. Mitchell, Richard Morris, C. A. Nelson; F. W. Owens, Helen B. Owens, G. E. Raynor, C. J. Rees, George Rosengarten, J. A. Roulton, J. A. Shohat, C. A. Shook, C. A. Short, L. L. Smail, W. M. Smith, Anna Pell Wheeler, A. H. Wilson, Clement Winston, W. L. Wright.

At the business meeting the following officers were chosen for next year: Chairman, J. R. Kline, University of Pennsylvania; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Kline, Caris, Owens, Raynor.

The next meeting will be held on Saturday, December 2, 1933, at Philadelphia.

The following papers were presented:

1. "Some boundary value problems in potential theory" by Professor G. E. Raynor, Lehigh University.

2. "The independent arcs of a continuous curve" by Professor J. R. Kline, University of Pennsylvania.

3. "On curves with assigned singularities" by Dr. Marguerite Lehr, Bryn Mawr College.

4. "The problem of measure" by Professor Orrin Frink, Jr., Pennsylvania State College.

5. "The life and work of Ramanujan" by Professor H. H. Mitchell, University of Pennsylvania.

Abstracts of the papers follow:

1. The first part of Professor Raynor's paper is a brief exposition of the methods which have proved most powerful in the recent progress made with the Dirichlet problem and gives some indication of the present status of the problem. The second part deals with the Dirichlet-Neumann problem, and presents necessary and sufficient conditions that the problem have a solution for the sphere with singular point at the center.

2. Professor Kline discussed the various definitions of a curve giving the short-comings of the earlier attempts to define this concept. He then discussed theorems concerning the Menger order of points of a curve and the independent arcs which join two points of a continuous curve. In particular he discussed the